Problems on Representation Theory from the 1997–2000 Tripos

2000, Paper B1, Question 6B.

(a) Let G be an *abelian* group, and V a vector space over an algebraically closed field k of arbitrary characteristic.

Suppose V is an irreducible representation of G. Show directly that $\dim_k V = 1$.

Show that the assumption that k is algebraically closed is necessary: give an example of a finite abelian group G acting irreducibly on a *real* vector space V with $\dim_{\mathbb{R}} V > 1$.

(b) Let G be a finite group, and suppose that every *complex* irreducible representation of G is one-dimensional. Show that G is abelian.

(c) Show that G is cyclic if and only if it has a faithful one-dimensional complex representation.

2000, Paper B2, Question 6B.

(a) Define $\wedge^2 V \subseteq V \otimes V$, and show that if a group G acts on V then $\wedge^2 V$ is a subrepresentation of $V \otimes V$.

(b) If χ_V is the character of V, what is the character of $\wedge^2 V$ in terms of χ_V?

(c) Let $G = \mathbf{S}_5$, the symmetric group on five letters, act on $V = \mathbb{C}^5$ by permuting coordinates. Decompose $\wedge^2 V$ into irreducible representations, and determine their characters.

2000, Paper B3, Question 5B.

Let $G = \mathrm{SU}_2$, and let V be a continuous representation of G on a finite dimensional vector space V.

(a) Define the character of V, χ_V, and show that $\chi_V \in \mathbb{N}[z, z^{-1}]$. [*You may assume results about the circle, but must state them clearly.*]

(b) Show that $\chi_V(z) = \chi_V(z^{-1})$.

(c) Let V be the irreducible four-dimensional representation of G. Decompose $V \otimes V$ into irreducible representations.

Hence decompose $\wedge^2 V$ into irreducible representations.

2000, Paper B4, Question 2B.

(a) Let $H \leq G$ be finite groups, and W a representation of H. Define the induced representation

$$\mathrm{Ind}_H^G W$$

and write the character of $\mathrm{Ind}_H^G W$ in terms of the character of W (you need not prove your answer).

(b) Let $G = \mathbf{D}_{2n}$ be the group of symmetries of the n-gon, $H = \mathbb{Z}/n\mathbb{Z}$ be the subgroup of rotations, and let ψ be a one dimensional character of H.

Suppose that $n = 2k+1$ is *odd*. Decompose $\mathrm{Ind}_H^G \psi$ into irreducibles for each such ψ, and hence list all the irreducible characters of G.

1999, Paper B1, Question 6C.

Let V be an irreducible complex representation of a finite group G, and let r be an integer ≥ 1.

(a) Define $\wedge^r V$, and show it is a representation of G;

(b) If $\lambda_1, \ldots, \lambda_d$ are the eigenvalues of $g \in G$ on V, where d is the dimension of V, find the eigenvalues of g on $\wedge^r V$. Describe how to determine the trace of g on $\wedge^r V$ from the characteristic polynomial of g on V.

(c) Let V be a three-dimensional irreducible representation of the alternating group $\mathbf{A}_4$. Compute the character of $\wedge^i V$, for $i = 1, 2, 3$.

1999, Paper B2, Question 6C.

Compute the character table of the symmetric group $\mathbf{S}_4$. Begin by listing the conjugacy classes and their orders.

Next computer the character table of the alternating group $\mathbf{A}_4$. For each irreducible character χ of $\mathbf{S}_4$, determine how the restriction of χ to $\mathbf{A}_4$ decomposes into irreducible characters.

1999, Paper B3, Question 5C.

Let G be a finite group, and suppose that G acts on finite sets X and Y. Define the permutation representation $\mathbb{C}[X]$, and compute its character χ. If ψ denotes the character of $\mathbb{C}[Y]$, show that (χ, ψ) is the number of G-orbits on $X \times Y$.

Hence (or otherwise) show that the symmetric group $\mathbf{S}_n$ has an irreducible complex representation of dimension equal to $n-1$.

Assume $n \geq 4$. Let $X = \big\{\{i,j\} \bigm| i \neq j, 1 \leq i,j \leq n\big\}$. Decompose $\mathbb{C}[X]$ as a representation of $\mathbf{S}_n$.

1999, Paper B4, Question 2C.

Write an essay on the representations of SU_2. Your essay should explain why every representation is a direct sum of irreducible representations, and include a list of irreducible representations, along with an explanation of why they are irreducible, and why they are the only irreducible representations.

1998, Paper B1, Question 1B.

(i) If V is a simple faithful module over the ring R and $D = \mathrm{Hom}_R(V,V)$, show that D is a division ring. Assuming that V is finite-dimensional over D, show further that $R = \mathrm{End}_D(V)$.

(ii) Let R be a commutative ring which, when considered as a module over itself, is a direct sum of a family of simple submodules. Show that R is a direct sum of fields.

1998, Paper 2, Question 1B.

Define the tensor product of two finite dimensional representation spaces for a finite group G, and show how to decompose $V \otimes V$ as the direct sum $\mathsf{S}^2(V) \oplus \wedge^2(V)$ of a symmetric and an alternating square.

Write down the characters of these two representations in terms of the character on V.

Let V be the four-dimensional irreducible representation of the symmetric group $\mathbf{S}_5$ contained in the natural permutation representation on the set $\{1,2,3,4,5\}$. Use the construction in the first part of the question to derive the character table of $\mathbf{S}_5$.

1998, Paper B3, Question 1B.

Let p be a prime number and H a normal subgroup of the finite group G of index p. If ρ is a one-dimensional representation of H, describe the p-dimensional induced representation $i_*\rho$ of G, and write down its character.

Let p be a prime and G be a non-abelian group of order p^3. Show that, up to equivalence, G has p^2 irreducible representations of dimension 1 and $p-1$ irreducible representations of dimension p.

Deduce that non-isomorphic groups can have the same character table.

1998, Paper B4, Question 1B.

Write an essay on the representations of the compact group SU_2. Your essay should include a description of the irreducible representations and a discussion of the Clebsch-Gordon relations.

1997, Paper B1, Question 1H.

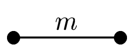
State the three Sylow theorems for a finite group G and prove the first of them.

If G is a group of order 48 prove that G contains either a normal subgroup of order 16 or a normal subgroup of order 8.

1997, Paper B2, Question 1H.

Explain carefully the meaning of the terms *reflection group*, *root system* and *Coxeter graph*, with its associated symmetric matrix A.

For a reflection group W of type $I_2(m)$, $m \geq 5$, with graph



determine the structure of W, the order of the root system Φ, and the value of $\det(2A)$. For which values of m is W crystallographic? Justify your answer.

1997, Paper B3, Question 1H.

Let $\mathbf{Q}_8$ be the quarternion group with presentation

$$\langle P, Q \mid P^4 = 1, P^2 = Q^2, QPQ^{-1} = P^{-1} \rangle,$$

and $\mathbf{C}_3$ the cyclic group of order 3 with generator X. Define an extension G of $\mathbf{Q}_8$ by $\mathbf{C}_3$ by means of the relations

$$XPX^{-1} = Q, XQX^{-1} = PQ, X(PQ)X^{-1} = P.$$

(a) Show that the subgroup H generated by P^2 is normal in G, and that G/H is isomorphic to the alternating group $\mathbf{A}_4$.

(b) Describe the character table of $\mathbf{A}_4$.

(c) Show that an irreducible representation of G which does not factor through $\mathbf{A}_4$ must have degree 2 over the complex numbers, and must be faithful.

(d) Using the fact that a faithful character of G restricts to faithful characters of the subgroups $\mathbf{Q}_8$ and $\mathbf{C}_3$, extend the character table of $\mathbf{A}_4$ to the character table of G.

1997, Paper B4, Question 1H.

(a) If G is a non-abelian group of prime-power order, show that G contains a normal abelian subgroup H which properly contains the centre $Z(G)$.

(b) Let E be an irreducible G-space over $\mathbb{C}$ affording a faithful representation of G. Show that there exists an H-direct sum decomposition

$$E = \bigoplus_{\psi} F_{\psi},$$

where F_{ψ} is the subspace of E spanned by all one-dimensional H-spaces with character ψ.

(c) Noting that $Z(G)$ is a proper subgroup of H and that E is faithful, show that $E \neq F_{\psi}$ for any one character ψ.

(d) Show that G permutes the subspaces F_{ψ} transitively.

(e) For some fixed character ψ_1, write $F = F_{\psi_1}$, and let H_1 be the stabilizer of F under the action of G. Show that $H \subseteq H_1$, that F is irreducible as an H_1-space, and that E is isomorphic to the induced representation space i_*F.

(f) What does (e) imply about the (not necessarily faithful) irreducible representations of G?