Solutions to Example Sheet 3

1. Let $G = \mathbf{A}_5$. For each pair of irreducible representations ρ, ρ',

(i) decompose $\rho \otimes \rho'$ into a sum of irreducible representations,

(ii) decompose $\bigwedge^2 \rho$ and $\mathsf{S}^2 \rho$ into irreducible representations,

(iii) decompose $\bigwedge^n \rho$ into irreducible representations, for $n \geq 3$,

(iv) check that if ρ is a non-trivial representation of $\mathbf{A}_5$ (hence faithful), every representation of $\mathbf{A}_5$ occurs with non-zero multiplicity in at least one of $\mathbb{1}, \rho, \rho^{\otimes 2}, \ldots$.

Solution. Straightforward but tedious. Use Problem **3**(iii) for $\bigwedge^n \rho$. The last part follows from Problem **11**. $\square$

2. (i) Let $\mathscr{C}_G$ be the space of class functions on G, and $\delta_{\mathcal{O}}$ be the basis of "delta functions on conjugacy classes". show that the functions $\delta_{\mathcal{O}}$ are orthogonal idempotents, and hence $\mathscr{C}_G$ is isomorphic as an algebra to a direct sum of copies of the one-dimensional algebra $\mathbb{C}$.

(ii) Let $G = \mathbb{Z}/n\mathbb{Z}$, and λ be the one-dimensional representation of G which sends $1 \in G$ to multiplication by $e^{2\pi i/n}$. show $\mathscr{C}_G = \mathbb{C}[\lambda]/(\lambda^n - 1)$. Check this is consistent with part (i).

(iii) Let $G = \mathbf{S}_3$, and ρ be the two-dimensional representation of G. Show $\mathscr{C}_G = \mathbb{C}[\rho]/(\rho^3 - \rho^2 - 2\rho)$.

Solution.

(i) Let $\mathcal{O}_1, \ldots, \mathcal{O}_n$ be the conjugacy classes of G. We define $\delta_{\mathcal{O}} : G \to \mathbb{C}$ by

$$\delta_{\mathcal{O}}(x) = \begin{cases} 1 & \text{if } x \in \mathcal{O}, \\ 0 & \text{if } x \notin \mathcal{O}. \end{cases}$$

Clearly $\delta_{\mathcal{O}}^2 = \delta_{\mathcal{O}}$ and so $\delta_{\mathcal{O}}$ is an idempotent element in the ring $\mathscr{C}_G$ (with respect to addition and multiplication of class functions). We have

$$\langle \delta_{\mathcal{O}_i}, \delta_{\mathcal{O}_j} \rangle = \frac{1}{|G|} \sum_{x \in G} \delta_{\mathcal{O}_i}(x) \overline{\delta_{\mathcal{O}_j}(x)} = \begin{cases} |\mathcal{O}_i|/|G| & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases} \tag{2.1}$$

since every $x \in G$ lies in a unique $\mathcal{O}_k$ for some $k \in \{1, \ldots, n\}$. Note that $|\mathcal{O}_i|/|G| \neq 0$ and so (2.1) shows that $\delta_{\mathcal{O}_1}, \ldots, \delta_{\mathcal{O}_n}$ are orthogonal to each other and in particular, linearly independent over $\mathbb{C}$. Given any $f \in \mathscr{C}_G$, f constant on conjugacy classes means that there exists $c_1, \ldots, c_n \in \mathbb{C}$ such that $f(x) = c_i$ for every $x \in \mathcal{O}_i$ and we have

$$f = c_1 \delta_{\mathcal{O}_1} + \cdots + c_n \delta_{\mathcal{O}_n}.$$

Hence $\{\delta_{\mathcal{O}_1}, \ldots, \delta_{\mathcal{O}_n}\}$ is a basis for $\mathscr{C}_G$. So as vector spaces over $\mathbb{C}$,

$$\mathscr{C}_G = \mathbb{C}\delta_{\mathcal{O}_1} \oplus \cdots \oplus \mathbb{C}\delta_{\mathcal{O}_n} \overset{\bigstar}{\cong} \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{n \text{ copies}} = \mathbb{C}^n.$$

Recall that $\mathbb{C}^n$ may be made into an algebra by defining product of $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in \mathbb{C}^n$ by

$$(a_1, \ldots, a_n)(b_1, \ldots, b_n) = (a_1 b_1, \ldots, a_n b_n). \tag{2.2}$$

Since $\delta_{\mathcal{O}_1}, \ldots, \delta_{\mathcal{O}_n}$ are idempotent and furthermore[1]

$$\delta_{\mathcal{O}_i} \delta_{\mathcal{O}_j} = \begin{cases} \delta_{\mathcal{O}_i} & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

So if $f = \sum a_i \delta_{\mathcal{O}_i}, g = \sum b_j \delta_{\mathcal{O}_j} \in \mathscr{C}_G$, we have

$$fg = \left(\sum_{i=1}^n a_i \delta_{\mathcal{O}_i} \right) \left(\sum_{j=1}^n b_j \delta_{\mathcal{O}_j} \right) = \sum_{i,j} a_i b_j \delta_{\mathcal{O}_i} \delta_{\mathcal{O}_j} = \sum_{i=1}^n a_i b_i \delta_{\mathcal{O}_i}. \tag{2.3}$$

Comparing (2.2) and (2.3), we see that $\bigstar$ is indeed an isomorphism of algebras.

[1] On second thoughts, this might be what the lecturer meant by 'orthogonal' idempotents. So maybe (2.1) is superfluous. But since it took me a while to TeX it, I'll leave it.

(ii) $\lambda : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^\times$, $1 \mapsto e^{2\pi i/n}$ is an irreducible character. Since $e^{2\pi i/n}$ is a primitive nth root of unity, $\mathbb{1}, \lambda, \dots, \lambda^{n-1}$ forms a complete set of irreducible characters of $\mathbb{Z}/n\mathbb{Z}$ (recall that $\widehat{\mathbb{Z}/n\mathbb{Z}} = \langle\lambda\rangle$ from the solution of Example Sheet **2**, Problem **7**(b)) and hence a basis of $\mathscr{C}_{\mathbb{Z}/n\mathbb{Z}}$. Now consider the map $\varphi : \mathbb{C}[x] \to \mathscr{C}_{\mathbb{Z}/n\mathbb{Z}}$, $f(x) \mapsto f(\lambda)$. φ is clearly a homomorphism of algebras. Since $1, x, \dots, x^{n-1}$ would be mapped to $\mathbb{1}, \lambda, \dots, \lambda^{n-1}$, so $\operatorname{rank}(\varphi) = \dim_{\mathbb{C}}(\mathscr{C}_{\mathbb{Z}/n\mathbb{Z}})$ and φ is surjective. We have $\mathbb{C}[x]/\ker(\varphi) \cong \operatorname{im}(\varphi) = \mathscr{C}_{\mathbb{Z}/n\mathbb{Z}}$. Since $\lambda^n = \mathbb{1}$, we have that $x^n - 1 \in \ker(\varphi)$ and so the $(x^n - 1) \subseteq \ker(\varphi)$. But

$$\dim_{\mathbb{C}}(\mathbb{C}[x]/(x^n - 1)) = n = \dim_{\mathbb{C}}(\mathscr{C}_{\mathbb{Z}/n\mathbb{Z}}) = \dim_{\mathbb{C}}(\mathbb{C}[x]/\ker(\varphi))$$

and so we must have equality $(x^n - 1) = \ker(\varphi)$. Therefore

$$\mathbb{C}[x]/(x^n - 1) \cong \mathscr{C}_{\mathbb{Z}/n\mathbb{Z}}.$$

(iii) It is easy to construct the character table of $\mathbf{S}_3$ — it has three conjugacy classes and its three irreducible characters are just the obvious ones: $\mathbb{1}$, $\varepsilon(\sigma) = \operatorname{sgn}(\sigma)$ and $\chi(\sigma) = |\{1,2,3\}^\sigma| - 1$. χ is the character of the two-dimensional representation ρ. The character table together with the characters corresponding to $\rho^{\otimes 2}$ and $\rho^{\otimes 3}$ is:

$\sigma^{\mathbf{S}_3}$	1	$(1\ 2)^{\mathbf{S}_3}$	$(1\ 2\ 3)^{\mathbf{S}_3}$
$\mathbb{1}$	1	1	1
ε	1	-1	1
χ	2	0	-1
χ^2	4	0	1
χ^3	8	0	-1

from which we can easily see that $\chi^2 = \mathbb{1} + \varepsilon + \chi$ and that $\chi^3 - \chi^2 - 2\chi = \mathbb{0}$. Since $\{\mathbb{1}, \varepsilon, \chi\}$ is a basis for $\mathscr{C}_{\mathbf{S}_3}$, then so is $\{\mathbb{1}, \chi, \chi^2\}$ by the first relation. Arguing as in the previous part, the map $\varphi : \mathbb{C}[x] \to \mathscr{C}_{\mathbf{S}_3}$, $f(x) \mapsto f(\chi)$ is surjective and $x^3 - x^2 - 2x \in \ker(\varphi)$ by the second relation. Again since

$$\dim_{\mathbb{C}}(\mathbb{C}[x]/(x^3 - x^2 - 2x)) = 3 = \dim_{\mathbb{C}}(\mathscr{C}_{\mathbf{S}_3}) = \dim_{\mathbb{C}}(\mathbb{C}[x]/\ker(\varphi)),$$

we have $\ker(\varphi) = (x^3 - x^2 - 2x)$ and

$$\mathbb{C}[x]/(x^3 - x^2 - 2x) \cong \mathscr{C}_{\mathbf{S}_3}.$$

□

Remark. Note that in Part (ii) and (iii) of this problem, all we need to arrive at the conclusion is the fact that the powers of λ and χ can be used to replace the irreducible characters as a basis for $\mathscr{C}_G$. By Problem **11**, every faithful character of G would have this property. So given such a character, we can follow the same procedure above to get the structure of $\mathscr{C}_G$ explicitly as a quotient of the polynomial algebra $\mathbb{C}[x]$.

3. Let V be a representation of G.

(i) Compute $\dim \mathsf{S}^n V$ and $\dim \bigwedge^n V$ for all n.

(ii) Let $g \in G$. Suppose g has eigenvalues $\lambda_1, \dots, \lambda_d$ on V. What are the eigenvalues of g on $\mathsf{S}^n V$ and $\bigwedge^n V$?

(iii) Let $f(x) = \det(g - xI)$ be the characteristic polynomial of g on V. Describe how to read $\operatorname{tr}(g, \bigwedge^n V)$ from the coefficients of $f(x)$.

(iv) ∗ Find a relation between $\operatorname{tr}(g, \mathsf{S}^n V)$ and the polynomial $f(x)$. (Hint: first do the case when $\dim V = 1$).

Solution.

(i) Let $d = \dim_{\mathbb{C}} V$. Then

$$\boxed{\dim_{\mathbb{C}} \mathsf{S}^n V = \binom{n+d-1}{n} \qquad \text{and} \qquad \dim_{\mathbb{C}} \textstyle\bigwedge^n V = \binom{d}{n}}$$

where we adopt the convention that $\binom{i}{j} = 0$ when $j > i$ (so $\dim_{\mathbb{C}} \bigwedge^n V = 0$ when $n > \dim_{\mathbb{C}} V$). Define the linear operators $S,\ A : V^{\otimes n} \to V^{\otimes n}$ by[2]

$$S = \frac{1}{n!} \sum_{\sigma \in \mathbf{S}_n} \sigma \qquad \text{and} \qquad A = \frac{1}{n!} \sum_{\sigma \in \mathbf{S}_n} \operatorname{sgn}(\sigma)\sigma.$$

Note that

$$\sigma S = S\sigma = S \qquad \text{and} \qquad \sigma A = A\sigma = \operatorname{sgn}(\sigma)A.$$

So for any $w \in V^{\otimes n}$, $\sigma(S(w)) = S(w)$ and $\sigma(A(w)) = \operatorname{sgn}(\sigma)A(w)$, which implies $S(V^{\otimes n}) \subseteq \mathsf{S}^n V$ and $A(V^{\otimes n}) \subseteq \bigwedge^n V$. Conversely, if $w \in \mathsf{S}^n V$, then $\sigma(w) = w$ and so

$$S(w) = \frac{1}{n!} \sum_{\sigma \in \mathbf{S}_n} \sigma(w) = \frac{1}{n!} \sum_{\sigma \in \mathbf{S}_n} (w) = w$$

implying that $w \in S(V^{\otimes n})$; likewise, if $w \in \bigwedge^n V$, then $\sigma(w) = \operatorname{sgn}(\sigma)w$ and so

$$A(w) = \frac{1}{n!} \sum_{\sigma \in \mathbf{S}_n} \operatorname{sgn}(\sigma)\sigma(w) = \frac{1}{n!} \sum_{\sigma \in \mathbf{S}_n} \operatorname{sgn}(\sigma)^2 w = \frac{1}{n!} \sum_{\sigma \in \mathbf{S}_n} w = w$$

implying that $w \in A(V^{\otimes n})$. Hence we have shown that $\mathsf{S}^n V = S(V^{\otimes n})$ and $\bigwedge^n V = A(V^{\otimes n})$.
Let $\{e_1, \dots, e_d\}$ be a basis of V. We adopt the following standard shorthand:

$$e_{i_1} \cdot \cdots \cdot e_{i_n} := S(e_{i_1} \otimes \cdots \otimes e_{i_n}) \qquad \text{and} \qquad e_{i_1} \wedge \cdots \wedge e_{i_n} := A(e_{i_1} \otimes \cdots \otimes e_{i_n}).$$

Since $S\sigma = S$, the term $e_{i_1} \cdot \cdots \cdot e_{i_n}$ depends only on the number of times each e_i enters this product and we can write

$$e_{i_1} \cdot \cdots \cdot e_{i_n} = e_1^{k_1} \cdot \cdots \cdot e_d^{k_d}$$

where k_i is the multiplicity of occurrence of e_i in $e_{i_1} \cdot \cdots \cdot e_{i_n}$. Since $A\sigma = \operatorname{sgn}(\sigma)A$, the term $e_{i_1} \wedge \cdots \wedge e_{i_n}$ changes sign under the permutation of any two factors $e_{i_\alpha} \leftrightarrow e_{i_\beta}$ and so $e_{i_1} \wedge \cdots \wedge e_{i_n} = 0$ if $e_{i_\alpha} = e_{i_\beta}$. In particular, $e_{i_1} \wedge \cdots \wedge e_{i_n} = 0$ whenever $n > d$.
The above discussion shows that

$$\mathscr{B}_1 = \{e_{i_1} \cdot \cdots \cdot e_{i_n} \mid 1 \le i_1 \le \cdots \le i_n \le d\} = \{e_1^{k_1} \cdot \cdots \cdot e_d^{k_d} \mid k_1 + \cdots + k_d = n\}$$

spans $\mathsf{S}^n V$ and

$$\mathscr{B}_2 = \{e_{i_1} \wedge \cdots \wedge e_{i_n} \mid 1 \le i_1 < \cdots < i_n \le d\}.$$

spans $\bigwedge^n V$.
Vectors in $\mathscr{B}_1$ are linearly independent: if $(k_1, \dots, k_d) \neq (l_1, \dots, l_d)$, then the tensors $e_1^{k_1} \cdot \cdots \cdot e_d^{k_d}$ and $e_1^{l_1} \cdot \cdots \cdot e_d^{l_d}$ are linear combinations of two non-intersecting subsets of basis elements of $V^{\otimes n}$. Likewise, vectors in $\mathscr{B}_2$ are linearly independent. Hence $\mathscr{B}_1$ and $\mathscr{B}_2$ are bases for $\mathsf{S}^n V$ and $\bigwedge^n V$ respectively.
The cardinality of $\mathscr{B}_1$ is $\binom{n+d-1}{n}$ = number of partitions of d into a sum of n non-negative integers. The cardinality of $\mathscr{B}_2$ is clearly $\binom{d}{n}$.

(ii) Recall that if G acts linearly on a vector space V, then the natural action on $V^{\otimes n}$ is given by

$$g(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}) = ge_{i_1} \otimes ge_{i_2} \otimes \cdots \otimes ge_{i_n}.$$

This action restricts to the G-invariant subspaces $\mathsf{S}^n V$ and $\bigwedge^n V$ as follows:

$$g(e_{i_1} \cdot \cdots \cdot e_{i_n}) = \frac{1}{n!} \sum_{\sigma \in \mathbf{S}_n} ge_{\sigma(i_1)} \otimes \cdots \otimes ge_{\sigma(i_n)},$$
$$g(e_{i_1} \wedge \cdots \wedge e_{i_n}) = \frac{1}{n!} \sum_{\sigma \in \mathbf{S}_n} \operatorname{sgn}(\sigma) ge_{\sigma(i_1)} \otimes \cdots \otimes ge_{\sigma(i_n)}.$$

[2] The $1/n!$ term is optional. It is there so that S and A are projections onto $\mathsf{S}^n V$ and $\bigwedge^n V$ respectively; also, S may then be interpreted as the 'average' of $\mathbf{S}_n$, ie. $\frac{1}{|G|} \sum_{g \in G} g$.

When $ge_i = \lambda_i e_i$ the expressions simplify as follows:

$$\begin{aligned} g(e_{i_1} \bullet \cdots \bullet e_{i_n}) &= \frac{1}{n!} \sum_{\sigma \in \mathbf{S}_n} \lambda_{\sigma(i_1)} \cdots \lambda_{\sigma(i_n)} e_{\sigma(i_1)} \otimes \cdots \otimes e_{\sigma(i_n)} \\ &= \lambda_{i_1} \cdots \lambda_{i_n} e_{i_1} \bullet \cdots \bullet e_{i_n}, \\ g(e_{i_1} \wedge \cdots \wedge e_{i_n}) &= \frac{1}{n!} \sum_{\sigma \in \mathbf{S}_n} \operatorname{sgn}(\sigma) \lambda_{\sigma(i_1)} \cdots \lambda_{\sigma(i_n)} e_{\sigma(i_1)} \otimes \cdots \otimes e_{\sigma(i_n)} \\ &= \lambda_{i_1} \cdots \lambda_{i_n} e_{i_1} \wedge \cdots \wedge e_{i_n} \end{aligned}$$

since $\lambda_{\sigma(i_1)} \cdots \lambda_{\sigma(i_n)} = \lambda_{i_1} \cdots \lambda_{i_n}$ for all $\sigma \in \mathbf{S}_n$. Hence, the set of eigenvalues of g on $\mathsf{S}^n V$ is

$$\{\lambda_{i_1} \cdots \lambda_{i_n} \mid 1 \leq i_1 \leq \cdots \leq i_n \leq d\} = \{\lambda_1^{k_1} \cdots \lambda_d^{k_d} \mid k_1 + \cdots + k_d = n\}$$

and the set of eigenvalues of g on $\bigwedge^n V$ is

$$\{\lambda_{i_1} \cdots \lambda_{i_n} \mid 1 \leq i_1 < \cdots < i_n \leq d\}.$$

(iii) Let $g \in G$. By (ii),

$$\operatorname{tr}(g, \textstyle\bigwedge^n V) = \displaystyle\sum_{1 \leq i_1 < \cdots < i_n \leq d} \lambda_{i_1} \cdots \lambda_{i_n}$$

and we have

$$f(x) = (\lambda_1 - x) \cdots (\lambda_d - x) = (-x)^d + \sum_{n=1}^{d} \operatorname{tr}(g, \textstyle\bigwedge^n V)(-x)^{d-n}.$$

So

$$\boxed{\operatorname{tr}(g, \textstyle\bigwedge^n V) = (-1)^{d-n} \times \text{coefficient of } x^{d-n} \text{ in } f(x)}$$

for $n = 1, \ldots, d$.

(iv) Let $g \in G$. By (ii),

$$\operatorname{tr}(g, \mathsf{S}^n V) = \sum_{k_1 + \cdots + k_d = n} \lambda_1^{k_1} \cdots \lambda_d^{k_d}$$

and we have

$$(1 + \lambda_1 x + \lambda_1^2 x^2 + \cdots) \cdots (1 + \lambda_d x + \lambda_d^2 x^2 + \cdots) = 1 + \sum_{n=1}^{\infty} \operatorname{tr}(g, \mathsf{S}^n V) x^n.$$

Noting that $(1 - \lambda_i x)^{-1} = 1 + \lambda_i x + \lambda_i^2 x^2 + \cdots$, we have

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} \operatorname{tr}(g, \mathsf{S}^n V) x^n &= \frac{1}{(1 - \lambda_1 x) \cdots (1 - \lambda_d x)} \\ &= \frac{1}{(-x)^d (\lambda_1 - 1/x) \cdots (\lambda_d - 1/x)} \\ &= \frac{1}{(-x)^d f(1/x)}. \end{aligned}$$

So

$$\boxed{\operatorname{tr}(g, \mathsf{S}^n V) = (-1)^d \times \text{coefficient of } x^{d+n} \text{ in power series expansion of } \frac{1}{f(1/x)}}$$

for $n = 1, 2, \ldots$.

□

Remark. Note that the two boxed formulas would have to be changed accordingly if you defined the characteristic polynomial as $f(x) = \det(xI - g)$.

4. Let G act on a finite set X, and $\mathbb{C}[X]$ be the permutation representation.

(i) Show
$$\mathsf{S}^2\,\mathbb{C}[X] = \mathbb{C}[\mathsf{S}^2\,X],$$
where $\mathsf{S}^2\,X$ is the set of orbits of $\mathbf{S}_2$ on $X \times X$. (Hence $\mathsf{S}^2\,X = \{A \subseteq X \mid |A| = 1 \text{ or } 2\}$.)

(ii) Compute the character of $\mathsf{S}^2\,\mathbb{C}[X]$.

(iii) Show that the trivial representation occurs in $\mathsf{S}^2\,\mathbb{C}[X]$ with multiplicity at least two. (Do not use Part (ii)!)

(iv) Show $\mathsf{S}^n\,\mathbb{C}[X] = \mathbb{C}[\mathsf{S}^n\,X]$.

(v) $\ast$ Can you find a description of $\bigwedge^2\mathbb{C}[X]$ analogous to the description of Part (i)? (You should use a non-trivial $\mathbf{S}_2$-equivariant vector bundle on $X \times X$ to replace the use of a permutation representation).

SOLUTION. Let $X = \{x_1, x_2, \ldots, x_d\}$ (note that we have implicitly ordered the elements in X by giving them integer indices; this is important in our solution). $\mathbf{S}_n$ acts on $X^n = X \times \cdots \times X$ by permuting coordinates. If $(x_{j_1}, x_{j_2}, \ldots, x_{j_n}) \in X^n$, we can certainly find a $\sigma \in \mathbf{S}_n$ such that $\sigma(x_{j_1}, x_{j_2}, \ldots, x_{j_n}) = (x_{i_1}, x_{i_2}, \ldots, x_{i_n})$ where the resulting indices satisfy $1 \leq i_1 \leq i_2 \leq \cdots \leq i_n \leq d$. Hence the set of orbits $\mathsf{S}^n\,X$ may be represented as
$$\mathsf{S}^n\,X = \{(x_{i_1}, \ldots, x_{i_n})^{\mathbf{S}_n} \mid 1 \leq i_1 \leq \cdots \leq i_n \leq d\}.$$
For notational simplicity, we write $x_i = i$ for $i = 1, \ldots, d$. When $n = 2$, we write:
$$\mathbb{C}[X] = \operatorname{span}_{\mathbb{C}}\{e_1, \ldots, e_d\} \qquad \mathsf{S}^2\,X = \{(i,j)^{\mathbf{S}_2} \mid 1 \leq i \leq j \leq d\}$$
$$\mathsf{S}^2\,\mathbb{C}[X] = \operatorname{span}_{\mathbb{C}}\{e_i \cdot e_j \mid 1 \leq i \leq j \leq d\} \qquad \mathbb{C}[\mathsf{S}^2\,X] = \operatorname{span}_{\mathbb{C}}\{e_{i,j} \mid 1 \leq i \leq j \leq d\}$$

(i) It is clear that $\mathsf{S}^2\,\mathbb{C}[X]$ and $\mathbb{C}[\mathsf{S}^2\,X]$ have the same dimension and are thus isomorphic as vector spaces. What we need to show is that they are isomorphic as G-modules. This follows from the G-equivariance of the linear map
$$\varphi : \mathsf{S}^2\,\mathbb{C}[X] \to \mathbb{C}[\mathsf{S}^2\,X]$$
$$e_i \cdot e_j \mapsto e_{i,j}$$
ie. $\varphi(ge_i \cdot e_j) = \varphi(e_{gi} \cdot e_{gj}) = e_{gi,gj} = ge_{i,j} = g\varphi(e_i \cdot e_j)$.

(ii) $\chi_{\mathsf{S}^2\,\mathbb{C}[X]}(g) = \chi_{\mathbb{C}[\mathsf{S}^2\,X]}(g) = |(\mathsf{S}^2\,X)^g|$.

(iii) This is obvious since $\mathsf{S}^2\,\mathbb{C}[X] = \mathbb{C}[\mathsf{S}^2\,X]$. Note that for any G-set S, the permutation representation $\mathbb{C}[S]$ always contains the trivial representation (so in this case, we just take $S = \mathsf{S}^2\,X$).

(iv) This follows from the G-equivariance of the map
$$\varphi : \mathsf{S}^n\,\mathbb{C}[X] \to \mathbb{C}[\mathsf{S}^n\,X]$$
$$e_{i_1} \cdot e_{i_2} \cdot \cdots \cdot e_{i_n} \mapsto e_{i_1, i_2, \ldots, i_n}$$
as above.

$\square$

5. For each irreducible representation ρ of $\mathbf{A}_4$, determine the character of $\operatorname{Ind}_{\mathbf{A}_4}^{\mathbf{S}_4}\rho$, and decompose this into irreducible representations. For each irreducible representation $\widetilde{\rho}$ of $\mathbf{S}_4$, decompose $\operatorname{Res}_{\mathbf{A}_4}^{\mathbf{S}_4}\widetilde{\rho}$ into irreducible representations. Check your answer is compatible with Frobenius reciprocity. Now do this for $\mathbf{S}_3 \subseteq \mathbf{S}_4$.

SOLUTION. Straightforward but tedious. $\square$

6. Determine the character table of the dihedral group $\mathbf{D}_{2n}$ of symmetries of the n-gon by inducing up representations of the rotation group $\mathbb{Z}/n\mathbb{Z} \hookrightarrow \mathbf{D}_{2n}$.

SOLUTION. Straightforward. $\square$

7. Let G be the Heisenberg group of order p^3. This is the subgroup

$$G = \left\{ \begin{pmatrix} 1 & a & x \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a, b, x \in \mathbb{F}_p \right\}$$

of matrices over the finite field $\mathbb{F}_p$, p a prime.

Let H be the subgroup of G of matrices with $a = 0$ and Z be the subgroup of G of matrices with $a = b = 0$.

(i) Show Z is the center of G, and that $G/Z = \mathbb{F}_p^2$. Note that this implies that the commutator subgroup $[G, G]$ is contained in Z. You can check by explicit computation that it equals Z, or if you are lazy you can deduce this from the list of irreducible representations, below.

(ii) Find all one dimensional representations of G.

(iii) Let $\psi : \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}^\times$ be a non-trivial one-dimensional representation of $\mathbb{F}_p$, and define a one-dimensional representation ρ_ψ of H by

$$\rho_\psi \left(\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \right) = \psi(x).$$

Show that $\operatorname{Ind}_H^G \rho_\psi$ is an irreducible representation of G.

(iv) Prove that the collection of representations constructed in Part (ii) and (iii) complete the list of all irreducible representations.

(v) Determine the character of the representations $\operatorname{Ind}_H^G \rho_\psi$.

Solution. Note that

$$g_1 g_2 = \begin{pmatrix} 1 & a_1 & x_1 \\ 0 & 1 & b_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_2 & x_2 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a_1+a_2 & x_1+x_2+a_1 b_2 \\ 0 & 1 & b_1+b_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

(i) Let $g_1 \in Z$. Then $g_1 g_2 = g_2 g_1$ for all $g_2 \in G$ iff $b_1 a_2 = a_1 b_2$ for all $a_2, b_2 \in \mathbb{F}_p$ iff $a_1 = b_1 = 0$. So Z is the center of G.

$$G/Z = \left\{ \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \;\middle|\; a, b \in \mathbb{F}_p \right\} \cong \mathbb{F}_p^2.$$

Since G/Z is abelian, $G' = [G, G]$ must be contained in Z. Now $[G : G'] = [G : Z][Z : G']$ and $[G : Z] = p^2$. So $[G : G'] = p^2$ ($[G : G'] = p^3$ is not possible since G is non-abelian) and thus $[Z : G'] = 1$, ie. $Z = G'$.

(ii) By Example Sheet **2**, Problem **7**, every one-dimensional representation of G is lifted from a representation of $G/G' = \mathbb{F}_p^2$. By Example Sheet **1**, Problem **9**, the representations of $\mathbb{F}_p^2$ are precisely

$$\chi_{l,m} : \mathbb{F}_p \times \mathbb{F}_p \to \mathbb{C}^\times, \quad (a, b) \mapsto e^{2(la+mb)\pi i/p}$$

for $l, m = 0, \ldots, p-1$, (ie. we let $\lambda = e^{2l\pi i/p}$ and $\mu = e^{2m\pi i/p}$ in Example Sheet **1**, Problem **9**). So the lifted one-dimensional representations of G are

$$\widetilde{\chi}_{l,m} : G \to \mathbb{C}^\times, \quad \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mapsto e^{2(la+mb)\pi i/p}$$

for $l, m = 0, \ldots, p-1$.

(iii) Let

$$\alpha = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \zeta = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$\begin{pmatrix} 1 & a & x \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \alpha^a \beta^b \zeta^x.$$

In particular $Z = \{\zeta^x \mid x = 0, \ldots, p-1\}$ and $H = \{\beta^b \zeta^x \mid b, x = 0, \ldots, p-1\}$.

Recall that ψ has the form $\psi_n : \mathbb{F}_p \to \mathbb{C}^\times$, $x \mapsto e^{2xn\pi i/p}$ for $n = 0, \ldots, p-1$. We will now compute the character of $\operatorname{Ind}_H^G \rho_\psi$.

<u>Case: $a = b = 0$.</u> Let $x \in \{0, \ldots, p-1\}$. Then $\zeta^x \in Z$. So

$$\chi_{\operatorname{Ind}_H^G \rho_\psi}(\zeta^x) = p\chi_{\rho_\psi}(\zeta^x) = p\psi(\zeta^x).$$

Case: $a = 0$, $b \neq 0$. Let $b \in \{1, \ldots, p-1\}$. Suppose β^b is conjugate to some $g \in G$, then $\beta^b Z$ is conjugate to gZ in the abelian group G/Z, so $\beta^b Z = gZ$, and therefore $g = \beta^b \zeta^x$ for some $x \in \{0, \ldots, p-1\}$. Since $\beta^b \notin Z$, the conjugacy class $(\beta^b)^G$ does not have size 1, and hence

$$(\beta^b)^G = \{\beta^b \zeta^x \mid x = 0, \ldots, p-1\}.$$

By the formula for character of induced representations, we have

$$\begin{aligned} \chi_{\operatorname{Ind}_H^G \rho_\psi}(\beta^b \zeta^x) &= \chi_{\rho_\psi}(\beta^b) + \chi_{\rho_\psi}(\beta^b \zeta) + \cdots + \chi_{\rho_\psi}(\beta^b \zeta^{p-1}) \\ &= \psi(\beta^b) + \psi(\beta^b \zeta) + \cdots + \psi(\beta^b \zeta^{p-1}) \\ &= \psi(\beta^b) \sum_{i=0}^{p-1} (\psi(\zeta))^i = 0 \end{aligned}$$

since $\psi(\zeta)$ is a pth root of unity.

Case: $a \neq 0$. In this case, $\alpha^a \beta^b \zeta^x \notin H$ and $(\alpha^a \beta^b \zeta^x)^G \cap H = \varnothing$, so

$$\chi_{\operatorname{Ind}_H^G \rho_\psi}(\alpha^a \beta^b \zeta^x) = 0$$

by the definition of induced characters.

Summarizing, if $\psi = \psi_n$, then

$$\chi_{\operatorname{Ind}_H^G \rho_\psi}(\alpha^a \beta^b \zeta^x) = \begin{cases} pe^{2xn\pi i/p} & \text{if } a = b = 0, \\ 0 & \text{otherwise,} \end{cases}$$

for $n = 0, \ldots, p-1$. $\operatorname{Ind}_H^G \rho_\psi$ is irreducible since

$$\langle \chi_{\operatorname{Ind}_H^G \rho_\psi}, \chi_{\operatorname{Ind}_H^G \rho_\psi} \rangle = \frac{1}{|G|} \sum_{g \in G} |\chi_{\operatorname{Ind}_H^G \rho_\psi}(g)|^2 = \frac{1}{p^3} \sum_{g \in Z} |\chi_{\operatorname{Ind}_H^G \rho_\psi}(g)|^2 = \frac{1}{p^3} \sum_{g \in Z} p^2 = 1.$$

(iv) The p^2 representations of dimension 1 in (ii), $\{\chi_{l,m} \mid l, m = 0, \ldots, p-1\}$, and the $p-1$ representations of dimension p in (iii), $\{\operatorname{Ind}_H^G \rho_{\psi_n} \mid n = 1, \ldots, p-1\}$, are all distinct, and the sum of the squares of their degrees is

$$p^2 \times 1^2 + (p-1) \times p^2 = p^3 = |G|$$

and so these complete the list of all irreducible representations.

(v) See (iii).

□

8. Suppose that V is a representation of G, and the character χ of V satisfies $\chi(g) \in \mathbb{R}$ for all $g \in G$. show that the trivial representation always occurs in $V \otimes V$. Conclude that if $\dim V > 2$, $V^{\otimes 2}$ is the sum of at least three irreducible representations.

Solution. Since $\chi(g)^2 \geq 0$,

$$\langle \mathbb{1}, \chi^2 \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)^2 \geq 0;$$

it cannot be 0 as $\chi(1) \neq 0$ and so the multiplicity of $\mathbb{1}$ in χ^2 must be non-zero.

If $\dim_{\mathbb{C}} V \geq 3$, then $\dim_{\mathbb{C}} \mathsf{S}^2 V \geq 6$ and $\dim_{\mathbb{C}} \bigwedge^2 V \geq 3$. We know that the trivial representation must occur in either $\mathsf{S}^2 V$ or $\bigwedge^2 V$ (or both). Since the trivial representation has dimension 1, either $\mathsf{S}^2 V$ or $\bigwedge^2 V$ (or both) must decompose further into two or more irreducible representations. In all cases, we would have at least three irreducible representations. □

9. Let (ρ, V) be an irreducible representation of a finite group G. How unique is the positive definite hermitian form on V?

10. Show that the number of irreducible characters of G taking only real values equals the number of conjugacy classes of elements g such that g^{-1} is conjugate to g.

SOLUTION. The character table of G may be viewed as an $n \times n$ matrix $A \in \mathrm{GL}(n, \mathbb{C})$ where n is the number of conjugacy classes of G (A is invertible by row or column orthogonality).

Observe that χ is an irreducible character if and only if $\overline{\chi}$ is. So $\overline{A} = \big(\overline{\chi}(g)\big)$ is simply a permutation of the rows of $A = \big(\chi(g)\big)$. If we let P denote the permutation matrix, then

$$PA = \overline{A}. \tag{10.1}$$

Since $\chi = \overline{\chi}$ precisely when $\chi(g) \in \mathbb{R}$ for all $g \in G$, the number of $\mathbb{R}$-valued characters equals the number of rows in A that are fixed under $A \mapsto \overline{A}$ which in turn equals the trace of P.

Now observe that $\chi(g^{-1}) = \overline{\chi(g)}$, so we may also view $\overline{A} = \big(\chi(g^{-1})\big)$ as a permutation of the columns of $A = \big(\chi(g)\big)$. Let Q denote the permutation matrix, then

$$AQ = \overline{A}. \tag{10.2}$$

Since $g^G = (g^{-1})^G$ precisely when the column corresponding to g^G equals that corresponding to $(g^{-1})^G$ (distinct conjugacy classes must assume different values on at least one irreducible character by column orthogonality), the number of conjugacy classes satisfying $g^G = (g^{-1})^G$ equals the number of columns that are fixed under $A \mapsto \overline{A}$ which in turn equals the trace of Q.

By (10.1) and (10.2), $AP = QA$, and the required result follows from

$$\mathrm{tr}(P) = \mathrm{tr}(APA^{-1}) = \mathrm{tr}(Q).$$

□

11. ∗ Let ρ be a faithful irreducible representation of G, with character χ, and suppose that χ takes t distinct values. Show that every irreducible representation of G occurs with non-zero multiplicity in at least one of

$$\mathbb{1}, \rho, \rho^{\otimes 2}, \ldots, \rho^{\otimes (t-1)}.$$

Hint: Let μ be an irreducible character, and suppose that $(\chi^j, \mu) = 0$ for $j = 0, \ldots, t-1$. Consider this an equation

$$(t \times t\text{-matrix of character of powers of } \rho)(\text{vector of } \mu) = 0$$

and invert the matrix to get a contradiction.

SOLUTION. Let $\alpha_0, \ldots, \alpha_{t-1} \in \mathbb{C}$ be the t distinct values in $\mathrm{im}(\chi)$ and let $G_i := \chi^{-1}(\alpha_j) = \{x \in G \mid \chi(x) = \alpha_i\}$ for $i = 0, \ldots, t-1$. We may assume $\alpha_0 = \chi(1)$. Since χ is faithful, $G_0 = \ker(\chi) = \{1\}$.

If μ is an irreducible character satisfying $(\chi^j, \mu) = 0$ for $j = 0, \ldots, t-1$, then

$$\sum\nolimits_{i=0}^{t-1} \alpha_i^j \Big(\sum\nolimits_{x \in G_i} \overline{\mu(x)}\Big) = |G| \cdot (\chi^j, \mu) = 0$$

for $j = 0, \ldots, t-1$. Let $\beta_i = \sum_{x \in G_i} \overline{\mu(x)}$ (note that $\beta_0 = \mu(1)$). Rewriting the system of linear equations in matrix form, we get

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha_0 & \alpha_1 & \cdots & \alpha_{t-1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_0^{t-1} & \alpha_1^{t-1} & \cdots & \alpha_{t-1}^{t-1} \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{t-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since the α_i's are all distinct and the Vandermonde matrix is invertible, we have $\beta_i = 0$ for all $i = 0, \ldots, t-1$. In particular, $0 = \beta_0 = \mu(1) \neq 0$ gives the required contradiction. □

Remark. The rows $\mathbf{v}_0, \ldots, \mathbf{v}_{t-1}$ of the Vandermonde matrix are linearly independent since a linear combination

$$a_0\mathbf{v}_0 + \cdots + a_{t-1}\mathbf{v}_{t-1} = \mathbf{0}$$

gives a polynomial $a_0 + a_1 X + \cdots + a_{t-1} X^{t-1}$ that has t distinct zeroes $1, \alpha_0, \ldots, \alpha_{t-1}$ and so must be the zero polynomial, ie. $a_0 = \cdots = a_{t-1} = 0$.

12. ✻ Prove the following theorem of Burnside. If ρ is an irreducible representation of G with character χ and $\dim \rho > 1$ then there exists a $g \in G$ such that $\chi(g) = 0$.

SOLUTION. (Sketch) Let ψ be a character (not necessarily irreducible) of the cyclic group $\mathbf{C}_n \cong \mathbb{Z}/n\mathbb{Z}$. Let $X = \{g \in \mathbf{C}_n \mid \langle g \rangle = \mathbf{C}_n\}$. We could show that $\prod_{g\in X} \psi(g)$ is an integer. Hence, if $\psi(g) \neq 0$, then $\prod_{g\in X} |\psi(g)|^2 \geq 1$ and so $\sum_{g\in X} |\psi(g)|^2 \geq |X|$ (by the fact that Arithmetic Mean $\geq$ Geometric Mean).

Now define an equivalence relation on the elements of G by $g \sim h$ iff $\langle g \rangle = \langle h \rangle$. Let $X_i = \{g \in G \mid \langle g \rangle = \langle g_i \rangle\}$, $i = 1, \dots, m$, be the equivalence classes. If $\chi(g) = \chi_\rho(g) \neq 0$ for all $g \in G$, then $\chi_{\mathrm{Res}^G_{\langle g_i \rangle} \rho}(g) \neq 0$ for all $g \in G$ and we may apply the above result to X_i and $\psi = \chi_{\mathrm{Res}^G_{\langle g_i \rangle} \rho}$ to get

$$\sum_{g\in X_i} |\psi(g)|^2 \geq |X_i|.$$

Let $g_1 = 1$. Then $X_1 = \{1\}$. So $\sum_{g\in X_1} |\psi(g)|^2 = |\psi(1)|^2 > 1 = |X_1|$ since $\dim \rho > 1$.

From $\langle \psi, \psi \rangle = 1$, we get

$$|G| = \sum_{g\in G} |\psi(g)|^2 = \sum_{i=1}^{m} \sum_{g\in X_i} |\psi(g)|^2 > \sum_{i=1}^{m} |X_i| = |G|,$$

which gives a contradiction.

Hence $\chi(g) = 0$ for some $g \in G$. □

13. Show that an abelian group has a faithful irreducible representation if and only if it is cyclic.

SOLUTION. We assume that the group G is finite and the representation ρ is over $\mathbb{C}$. Recall that complex irreducible representations of abelian groups are one-dimensional (Example Sheet **1**, Problem **9**) and that each $\rho(x) \in \mathbb{C}^\times$ is an Nth root of unity where $N = |G|$ (Example Sheet **1**, Problem **7**). So $\rho(x) \in \partial\mathbb{D} = \{z \in \mathbb{C} \mid |z| = 1\}$ for all $x \in G$.

If G is abelian and $\rho : G \to \mathbb{C}^\times$ is faithful, then $G \cong \rho(G)$. But $\rho(G) \leq \partial\mathbb{D} \cong S^1$ and since the only finite subgroups of S^1 are either dihedral or cyclic, we must have G is cyclic as it is abelian.

If $G = \langle x \rangle$ is a cyclic group, then $\rho : G \to \mathbb{C}^\times$, $x^n \mapsto e^{2n\pi i/N}$ is clearly a faithful representation. □

14. Let $\Gamma_r = \{A \subseteq \{1, \dots, n\} \mid |A| = r\}$, and χ_r be the character of the permutation representation $\mathbb{C}[\Gamma_r]$ of $\mathbf{S}_n$. Compute (χ_r, χ_s) and hence decompose $\mathbb{C}[\Gamma_r]$ as a representation of $\mathbf{S}_n$.

SOLUTION. (Impetus for this solution provided by Dominic Pinto, Caius College) By Problem **9** in Example Sheet **2**,

$$(\chi_r, \chi_s) = \#\mathbf{S}_n\text{-orbits on } \Gamma_r \times \Gamma_s.$$

Let $\Delta := \{(x, x) \mid x \in X\}$. Recall that $\mathbf{S}_n$ acts transitively on the $\mathbf{S}_n$-stable subsets Δ and $X \times X \setminus \Delta$. It follows that the $\mathbf{S}_n$-orbits on $\Gamma_r \times \Gamma_s$ are

$$\mathcal{O}_i = \{A \times B \in \Gamma_r \times \Gamma_s \mid |A \cap B| = i\}. \tag{14.1}$$

for $i = \max(0, s + r - n), \dots, \min(s, r)$ (a moment's thought would show that these are the only possible values i could take). Hence

$$(\chi_r, \chi_s) = \min(s, r) - \max(0, s + r - n) + 1. \tag{14.2}$$

Note that (14.1) also follows from a direct argument: if $A\times B$, $C\times D \in \Gamma_r\times\Gamma_s$ and $|A\cap B| = i = |C\cap D|$, let

$$A \cap B = \{x_1, \dots, x_i\}, \quad A \setminus B = \{x_{i+1}, \dots, x_r\}, \quad B \setminus A = \{x'_{i+1}, \dots, x'_s\}, \quad X \setminus (A \cup B) = \{x''_{r+s-i+1}, \dots, x''_n\},$$
$$C \cap D = \{y_1, \dots, y_i\}, \quad C \setminus D = \{y_{i+1}, \dots, y_r\}, \quad D \setminus C = \{y'_{i+1}, \dots, y'_s\}, \quad X \setminus (C \cup D) = \{y''_{r+s-i+1}, \dots, y''_n\}.$$

Then the permutation in $\mathbf{S}_n$ defined by

$$\begin{pmatrix} x_1 & \cdots & x_i & x_{i+1} & \cdots & x_r & x'_{i+1} & \cdots & x'_s & x''_{r+s-i+1} & \cdots & x''_n \\ \downarrow & & \downarrow & \downarrow & & \downarrow & \downarrow & & \downarrow & \downarrow & & \downarrow \\ y_1 & \cdots & y_i & y_{i+1} & \cdots & y_r & y'_{i+1} & \cdots & y'_s & y''_{r+s-i+1} & \cdots & y''_n \end{pmatrix}$$

would send $A \times B$ to $C \times D$, ie. they lie in the same $\mathbf{S}_n$-orbit. Since $\mathbf{S}_n$ permutes elements in Δ, $A \times B$ and $C \times D$ with $|A \cap B| \neq |C \cap D|$ cannot lie in the same $\mathbf{S}_n$-orbit.

To decompose $\mathbb{C}[\Gamma_r]$ into irreducibles, observe that

① $\Gamma_1 = \{1, 2, \dots n\}$, so $\mathbb{C}[\Gamma_1]$ is just the usual permutation representation of $\mathbf{S}_n$ which we know breaks up into exactly two representations: the trivial representation $\mathbb{1}$ and an $(n-1)$-dimensional irreducible representation V_1 (see your lecture notes).

② Γ_n is a singleton, so $\mathbb{C}[\Gamma_n]$ is the trivial representation. Hence $\chi_n = \mathbb{1}$.

③ $\mathbb{C}[\Gamma_r]$ and $\mathbb{C}[\Gamma_{n-r}]$ are equivalent representations of $\mathbf{S}_n$. The obvious G-isomorphism being $\mathbb{C}[\Gamma_r] \to \mathbb{C}[\Gamma_{n-r}]$, $A \mapsto X \setminus A$ for $A \in \Gamma_r$. G-equivariance follows from $\sigma(X \setminus A) = X \setminus \sigma A$ for $\sigma \in \mathbf{S}_n$. Hence $\chi_r = \chi_{n-r}$.

By ②, (14.2) with $s = n$ gives

$$(\chi_r, \mathbb{1}) = (\chi_r, \chi_n) = 1.$$

By ③, we may assume that $s \le r \le n/2$, so $s + r \le n$ and (14.2) gives

$$(\chi_r, \chi_s) = s + 1.$$

For $r = 2$, $(\chi_2, \mathbb{1}) = 1$ and $(\chi_2, \chi_1) = 2$ imply $\mathbb{1}$ and V_1 both occur with multiplicity 1 in $\mathbb{C}[\Gamma_2]$. Since $(\chi_2, \chi_2) = 3$, there is a third irreducible representation V_2 of dimension $\deg(\chi_2) - \deg(\chi_1) = \binom{n}{2} - \binom{n}{1}$. In general, since $(\chi_r, \chi_{r-1}) = r$ and $(\chi_r, \chi_r) = r + 1$, $\mathbb{C}[\Gamma_r]$ contains an irreducible representation V_r of degree $\deg(\chi_r) - \deg(\chi_{r-1}) = \binom{n}{r} - \binom{n}{r-1}$ that is not in $\mathbb{C}[\Gamma_{r-1}]$. Inductively we get

$$\mathbb{C}[\Gamma_r] = \mathbb{1} \oplus V_1 \oplus \cdots \oplus V_r.$$

□

15. Using Problems 14 and 4, decompose $\mathsf{S}^m \mathbb{C}^n$ as representation of $\mathbf{S}_n$ for all $m \ge 1$.

Solution. Let $X = \{1, 2, \dots, n\}$. Note that $\mathbb{C}[X] \cong \mathbb{C}^n$ as $\mathbf{S}_n$-modules.

$$\begin{aligned}
\mathsf{S}^m \mathbb{C}^n &= \mathsf{S}^m \mathbb{C}[X] = \mathbb{C}[\mathsf{S}^m X] \\
&= \mathbb{C}[\Gamma_m] \oplus \mathbb{C}[\Gamma_{m-1}] \oplus \cdots \oplus \mathbb{C}[\Gamma_1] \\
&= (\mathbb{1} \oplus V_1 \oplus \cdots \oplus V_m) \oplus (\mathbb{1} \oplus V_1 \oplus \cdots \oplus V_{m-1}) \oplus \cdots \oplus (\mathbb{1} \oplus V_1) \\
&= m\mathbb{1} \oplus (m-1)V_1 \oplus \cdots \oplus V_n
\end{aligned}$$

□

16. Let V be an irreducible representation of G, with character χ. Show that $|G|^{-1} \sum_{g \in G} \chi(g^2)$ is either 0, 1 or -1, and that this zero precisely when χ is not *real* valued, ie. when V is not isomorphic to V^*. (Hint: consider $V \otimes V = \mathsf{S}^2 V \oplus \bigwedge^2 V$).

Solution. Since χ is irreducible, it occurs with multiplicity one in itself and if it is not real valued, the $\overline{\chi}$ is a different irreducible character, ie.

$$\langle \chi, \overline{\chi} \rangle = \begin{cases} 1 & \text{if } \chi \text{ is real valued,} \\ 0 & \text{if } \chi \text{ is not real valued.} \end{cases}$$

Since

$$\langle \chi, \overline{\chi} \rangle = \frac{1}{|G|} \sum_{g \in G} \chi^2(g) = \langle \chi^2, \mathbb{1} \rangle = \langle \chi_{\mathsf{S}^2 V} + \chi_{\wedge^2 V}, \mathbb{1} \rangle = \langle \chi_{\mathsf{S}^2 V}, \mathbb{1} \rangle + \langle \chi_{\wedge^2 V}, \mathbb{1} \rangle,$$

we have the following possibilities

① $\langle \chi_{\mathsf{S}^2 V}, \mathbb{1} \rangle = 1$ and $\langle \chi_{\wedge^2 V}, \mathbb{1} \rangle = 0$ when χ is real valued,

② $\langle \chi_{\mathsf{S}^2 V}, \mathbb{1} \rangle = 0$ and $\langle \chi_{\wedge^2 V}, \mathbb{1} \rangle = 0$ when χ is *not* real valued,

③ $\langle \chi_{\mathsf{S}^2 V}, \mathbb{1} \rangle = 0$ and $\langle \chi_{\wedge^2 V}, \mathbb{1} \rangle = 1$ when χ is real valued.

Hence this gives

$$\frac{1}{|G|} \sum_{g \in G} \chi(g^2) = \langle \chi_{\mathsf{S}^2 V}, \mathbb{1} \rangle - \langle \chi_{\wedge^2 V}, \mathbb{1} \rangle = \begin{cases} 1 & \text{if ① holds,} \\ 0 & \text{if ② holds,} \\ -1 & \text{if ③ holds.} \end{cases}$$

Note that ② happens precisely when χ is not real valued. □

17. Let $G \leq \mathrm{GL}_n(\mathbb{C})$, G a finite group. Show that $\sum_{g\in G} \operatorname{tr}(g) = 0$ implies $\sum_{g\in G} g = 0$.

SOLUTION. We may regard $G \hookrightarrow \mathrm{GL}_n(\mathbb{C})$, $g \mapsto g$ as a representation of G. Recall that every representation of a finite group is equivalent to a unitary representation (the proof is similar to that in Example Sheet **1**, Problem **8**(i)). In this case, it means that we may assume that $G \leq \mathrm{U}(n)$, ie. $g^* := \overline{g}^T = g^{-1}$ for all $g \in G$. Let $A = \sum_{g\in G} g \in \mathrm{M}_n(\mathbb{C})$. Observe that

① A is self-adjoint (a.k.a. hermitian) since

$$A^* = \sum_{g\in G} g^* = \sum_{g\in G} g^{-1} = \sum_{g\in G} g = A.$$

② For all $x \in G$,

$$xA = \sum_{g\in G} xg = \sum_{g\in G} g = A.$$

Suppose $A \neq 0$. Then by ①, $A^2 = AA^*$ is positive definite and so has eigenvalues $\lambda_1, \dots, \lambda_n \in (0, \infty)$. Since

$$A^2 = \Big(\sum_{g\in G} g\Big) A = \sum_{g\in G} gA \overset{②}{=} \sum_{g\in G} A = |G|A,$$

we get $\operatorname{tr}(A^2) = |G| \operatorname{tr}(A)$ and $\operatorname{tr}(A^2) = \lambda_1 + \cdots + \lambda_n > 0$ would then imply that $\operatorname{tr}(A) > 0$. $\square$

Remark. Given any $G \leq \mathrm{GL}_n(\mathbb{C})$, $G \hookrightarrow \mathrm{GL}_n(\mathbb{C})$ is often known as the *natural* representation of G.