Solutions to Example Sheet 3

1. Let \( G = A_5 \). For each pair of irreducible representations \( \rho, \rho' \),
   
   (i) decompose \( \rho \otimes \rho' \) into a sum of irreducible representations,
   
   (ii) decompose \( \wedge^2 \rho \) and \( \wedge^2 \rho \) into irreducible representations,
   
   (iii) decompose \( \wedge^n \rho \) into irreducible representations, for \( n \geq 3 \),
   
   (iv) check that if \( \rho \) is a non-trivial representation of \( A_5 \) (hence faithful), every representation of \( A_5 \) occurs with non-zero multiplicity in at least one of \( 1, \rho, \rho \otimes 2, \ldots \).

   **SOLUTION.** Straightforward but tedious. Use Problem 3(iii) for \( \wedge^n \rho \). The last part follows from Problem 11. \( \Box \)

2. (i) Let \( \mathcal{C}_G \) be the space of class functions on \( G \), and \( \delta_G \) be the basis of “delta functions on conjugacy classes”, show that the functions \( \delta_G \) are orthogonal idempotents, and hence \( \mathcal{C}_G \) is isomorphic as an algebra to a direct sum of copies of the one-dimensional algebra \( \mathbb{C} \).

   (ii) Let \( G = \mathbb{Z}/n\mathbb{Z} \), and \( \lambda \) be the one-dimensional representation of \( G \) which sends 1 \( \in G \) to multiplication by \( e^{2\pi i/n} \). Show \( \mathcal{C}_G = \mathbb{C}[\lambda]/(\lambda^n - 1) \). Check this is consistent with part (i).

   (iii) Let \( G = S_3 \), and \( \rho \) be the two-dimensional representation of \( G \). Show \( \mathcal{C}_G = \mathbb{C}[\rho]/(\rho^3 - \rho^2 - 2\rho) \).

   **SOLUTION.**

   (i) Let \( \mathcal{O}_1, \ldots, \mathcal{O}_n \) be the conjugacy classes of \( G \). We define \( \delta_G : G \to \mathbb{C} \) by

   \[
   \delta_G(x) = \begin{cases} 
   1 & \text{if } x \in \mathcal{O}, \\
   0 & \text{if } x \notin \mathcal{O}.
   \end{cases}
   \]

   Clearly \( \delta^2_G = \delta_G \) and so \( \delta_G \) is an idempotent element in the ring \( \mathcal{C}_G \) (with respect to addition and multiplication of class functions). We have

   \[
   \langle \delta_{\mathcal{O}_i}, \delta_{\mathcal{O}_j} \rangle = \frac{1}{|G|} \sum_{x \in G} \delta_{\mathcal{O}_i}(x) \overline{\delta_{\mathcal{O}_j}(x)} = \begin{cases} 
   |\mathcal{O}_i|/|G| & \text{if } i = j, \\
   0 & \text{if } i \neq j
   \end{cases}
   \]

   (2.1) since every \( x \in G \) lies in a unique \( \mathcal{O}_k \) for some \( k \in \{1, \ldots, n\} \). Note that \( |\mathcal{O}_i|/|G| \neq 0 \) and so (2.1) shows that \( \delta_{\mathcal{O}_1}, \ldots, \delta_{\mathcal{O}_n} \) are orthogonal to each other and in particular, linearly independent over \( \mathbb{C} \). Given any \( f \in \mathcal{C}_G \), \( f \) constant on conjugacy classes means that there exists \( c_1, \ldots, c_n \in \mathbb{C} \) such that \( f(x) = c_i \) for every \( x \in \mathcal{O}_i \), and we have

   \[
   f = c_1 \delta_{\mathcal{O}_1} + \cdots + c_n \delta_{\mathcal{O}_n}.
   \]

   Hence \( \{\delta_{\mathcal{O}_1}, \ldots, \delta_{\mathcal{O}_n}\} \) is a basis for \( \mathcal{C}_G \). So as vector spaces over \( \mathbb{C} \),

   \[
   \mathcal{C}_G = \mathbb{C}\delta_{\mathcal{O}_1} \oplus \cdots \oplus \mathbb{C}\delta_{\mathcal{O}_n} \cong \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{n \text{ copies}} = \mathbb{C}^n.
   \]

   Recall that \( \mathbb{C}^n \) may be made into an algebra by defining product of \( (a_1, \ldots, a_n), (b_1, \ldots, b_n) \in \mathbb{C}^n \) by

   \[
   (a_1, \ldots, a_n)(b_1, \ldots, b_n) = (a_1b_1, \ldots, a_nb_n).
   \]

   (2.2) Since \( \delta_{\mathcal{O}_1}, \ldots, \delta_{\mathcal{O}_n} \) are idempotent and furthermore\(^1\)

   \[
   \delta_{\mathcal{O}_i}\delta_{\mathcal{O}_j} = \begin{cases} 
   \delta_{\mathcal{O}_i} & \text{if } i = j, \\
   0 & \text{if } i \neq j.
   \end{cases}
   \]

   So if \( f = \sum a_i\delta_{\mathcal{O}_i}, \ g = \sum b_j\delta_{\mathcal{O}_j} \in \mathcal{C}_G \), we have

   \[
   fg = \left(\sum_{i=1}^n a_i\delta_{\mathcal{O}_i}\right)\left(\sum_{j=1}^n b_j\delta_{\mathcal{O}_j}\right) = \sum_{i,j} a_ib_j\delta_{\mathcal{O}_i}\delta_{\mathcal{O}_j} = \sum_{i=1}^n a_i b_i \delta_{\mathcal{O}_i}.
   \]

   (2.3)

   Comparing (2.2) and (2.3), we see that \( \star \) is indeed an isomorphism of algebras.

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\(^1\)On second thoughts, this might be what the lecturer meant by ‘orthogonal’ idempotents. So maybe (2.1) is superfluous. But since it took me a while to \( \LaTeX \) it, I’ll leave it.
(ii) \( \lambda : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^\times \), \( 1 \mapsto e^{2\pi i/n} \) is an irreducible character. Since \( e^{2\pi i/n} \) is a primitive \( n \)-th root of unity, \( 1, \lambda, \ldots, \lambda^{n-1} \) forms a complete set of irreducible characters of \( \mathbb{Z}/n\mathbb{Z} \) (recall that \( \mathbb{Z}/n\mathbb{Z} = \langle \lambda \rangle \) from the solution of Example Sheet 2, Problem 7(b)) and hence a basis of \( \mathcal{C}_{\mathbb{Z}/n\mathbb{Z}} \). Now consider the map \( \varphi : \mathbb{C}[x] \to \mathcal{C}_{\mathbb{Z}/n\mathbb{Z}}, \ f(x) \mapsto f(\lambda) \). \( \varphi \) is clearly a homomorphism of algebras. Since \( 1, x, \ldots, x^{n-1} \) would be mapped to \( 1, \lambda, \ldots, \lambda^{n-1} \), so rank(\( \varphi \)) = dim\( \mathbb{C}[x]/\mathcal{C}_{\mathbb{Z}/n\mathbb{Z}} \) and \( \varphi \) is surjective. We have \( \mathbb{C}[x]/\ker(\varphi) \cong \text{im}(\varphi) = \mathcal{C}_{\mathbb{Z}/n\mathbb{Z}} \). Since \( \lambda^n = 1 \), we have that \( x^n - 1 \in \ker(\varphi) \) and so the \( (x^n - 1) \subseteq \ker(\varphi) \). But
\[
\dim_{\mathbb{C}}(\mathbb{C}[x]/(x^n - 1)) = n = \dim_{\mathbb{C}}(\mathcal{C}_{\mathbb{Z}/n\mathbb{Z}}) = \dim_{\mathbb{C}}(\mathbb{C}[x]/\ker(\varphi))
\]
and so we must have equality \( (x^n - 1) = \ker(\varphi) \). Therefore
\[
\mathbb{C}[x]/(x^n - 1) \cong \mathcal{C}_{\mathbb{Z}/n\mathbb{Z}}.
\]

(iii) It is easy to construct the character table of \( S_3 \) — it has three conjugacy classes and its three irreducible characters are just the obvious ones: \( 1, \varepsilon, \chi \). \( \varepsilon(\sigma) = \text{sgn}(\sigma) \) and \( \chi(\sigma) = |\{1, 2, 3\}^\sigma| - 1 \). \( \chi \) is the character of the two-dimensional representation \( \rho \). The character table together with the characters corresponding to \( \rho^\oplus 2 \) and \( \rho^\oplus 3 \) is:

\[
\begin{array}{ccc}
\sigma & 1 & (1 2)_{S_3} & (1 2 3)_{S_3} \\
1 & 1 & 1 & 1 \\
\varepsilon & 1 & -1 & 1 \\
\chi & 2 & 0 & -1 \\
\chi^2 & 4 & 0 & 1 \\
\chi^3 & 8 & 0 & -1 \\
\end{array}
\]

from which we can easily see that \( \chi^2 = 1 + \varepsilon + \chi \) and that \( \chi^3 - \chi^2 - 2\chi = 0 \). Since \( \{1, \varepsilon, \chi\} \) is a basis for \( \mathcal{C}_{S_3} \), then so is \( \{1, \chi, \chi^2\} \) by the first relation. Arguing as in the previous part, the map \( \varphi : \mathbb{C}[x] \to \mathcal{C}_{S_3}, \ f(x) \mapsto f(\chi) \) is surjective and \( x^3 - x^2 - 2x \in \ker(\varphi) \) by the second relation. Again since
\[
\dim_{\mathbb{C}}(\mathbb{C}[x]/(x^3 - x^2 - 2x)) = 3 = \dim_{\mathbb{C}}(\mathcal{C}_{S_3}) = \dim_{\mathbb{C}}(\mathbb{C}[x]/\ker(\varphi)),
\]
we have \( \ker(\varphi) = (x^3 - x^2 - 2x) \) and
\[
\mathbb{C}[x]/(x^3 - x^2 - 2x) \cong \mathcal{C}_{S_3}.
\]

**Remark.** Note that in Part (ii) and (iii) of this problem, all we need to arrive at the conclusion is the fact that the powers of \( \lambda \) and \( \chi \) can be used to replace the irreducible characters as a basis for \( \mathcal{C}_G \). By Problem 11, every faithful character of \( G \) would have this property. So given such a character, we can follow the same procedure above to get the structure of \( \mathcal{C}_G \) explicitly as a quotient of the polynomial algebra \( \mathbb{C}[x] \).

3. Let \( V \) be a representation of \( G \).

(i) Compute \( \dim S^n V \) and \( \dim \bigwedge^n V \) for all \( n \).

(ii) Let \( g \in G \). Suppose \( g \) has eigenvalues \( \lambda_1, \ldots, \lambda_d \) on \( V \). What are the eigenvalues of \( g \) on \( S^n V \) and \( \bigwedge^n V \) ?

(iii) Let \( f(x) = \det(g - xI) \) be the characteristic polynomial of \( g \) on \( V \). Describe how to read \( \text{tr}(g, \bigwedge^n V) \) from the coefficients of \( f(x) \).

(iv) Find a relation between \( \text{tr}(g, S^n V) \) and the polynomial \( f(x) \). (Hint: first do the case when \( \dim V = 1 \).

**Solution.**

(i) Let \( d = \dim_{\mathbb{C}} V \). Then

\[
\dim_{\mathbb{C}} S^n V = \binom{n + d - 1}{n} \quad \text{and} \quad \dim_{\mathbb{C}} \bigwedge^n V = \binom{d}{n}
\]
where we adopt the convention that \( \binom{j}{i} = 0 \) when \( j > i \) (so \( \dim \bigwedge^n V = 0 \) when \( n > \dim V \)).

Define the linear operators \( S, A : V^\otimes n \to V^\otimes n \) by

\[
S = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma \quad \text{and} \quad A = \frac{1}{n!} \sum_{\sigma \in S_n} \sgn(\sigma) \sigma.
\]

Note that

\[
\sigma S = S \sigma = S \quad \text{and} \quad \sigma A = A \sigma = \sgn(\sigma) A.
\]

So for any \( w \in V^\otimes n \), \( \sigma(S(w)) = S(\sigma(w)) \) and \( \sigma(A(w)) = \sgn(\sigma) A(\sigma(w)) \), which implies \( S(V^\otimes n) \subseteq S^n V \) and \( A(V^\otimes n) \subseteq \bigwedge^n V \). Conversely, if \( w \in S^n V \), then \( \sigma(w) = w \) and so

\[
S(w) = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma(w) = \frac{1}{n!} \sum_{\sigma \in S_n} (w) = w
\]

implying that \( w \in S(V^\otimes n) \); likewise, if \( w \in \bigwedge^n V \), then \( \sigma(w) = \sgn(\sigma) w \) and so

\[
A(w) = \frac{1}{n!} \sum_{\sigma \in S_n} \sgn(\sigma) \sigma(w) = \frac{1}{n!} \sum_{\sigma \in S_n} \sgn(\sigma)^2 w = \frac{1}{n!} \sum_{\sigma \in S_n} w = w
\]

implying that \( w \in A(V^\otimes n) \). Hence we have shown that \( S^n V = S(V^\otimes n) \) and \( \bigwedge^n V = A(V^\otimes n) \).

Let \( \{e_1, \ldots, e_d\} \) be a basis of \( V \). We adopt the following standard shorthand:

\[
e_{i_1} \cdots e_{i_n} := S(e_{i_1} \otimes \cdots \otimes e_{i_n}) \quad \text{and} \quad e_{i_1} \wedge \cdots \wedge e_{i_n} := A(e_{i_1} \otimes \cdots \otimes e_{i_n}).
\]

Since \( S \sigma = S \), the term \( e_{i_1} \cdots e_{i_n} \) depends only on the number of times each \( e_i \) enters this product and we can write

\[
e_{i_1} \cdots e_{i_n} = e_{i_1}^{k_1} \cdots e_{i_d}^{k_d}
\]

where \( k_i \) is the multiplicity of occurrence of \( e_i \) in \( e_{i_1} \cdots e_{i_n} \). Since \( A \sigma = \sgn(\sigma) A \), the term \( e_{i_1} \wedge \cdots \wedge e_{i_n} \) changes sign under the permutation of any two factors \( e_{i_i} \leftrightarrow e_{i_j} \) and so \( e_{i_1} \wedge \cdots \wedge e_{i_n} = 0 \) if \( e_{i_i} = e_{i_j} \). In particular, \( e_{i_1} \wedge \cdots \wedge e_{i_n} = 0 \) whenever \( n > d \).

The above discussion shows that

\[
\mathcal{B}_1 = \{e_{i_1} \cdots e_{i_n} \mid 1 \leq i_1 \leq \cdots \leq i_n \leq d\} = \{e_1^{k_1} \cdots e_d^{k_d} \mid k_1 + \cdots + k_d = n\}
\]

spans \( S^n V \) and

\[
\mathcal{B}_2 = \{e_{i_1} \wedge \cdots \wedge e_{i_n} \mid 1 \leq i_1 < \cdots < i_n \leq d\}.
\]

spans \( \bigwedge^n V \).

Vectors in \( \mathcal{B}_1 \) are linearly independent: if \( (k_1, \ldots, k_d) \neq (l_1, \ldots, l_d) \), then the tensors \( e_1^{k_1} \cdots e_d^{k_d} \) and \( e_1^{l_1} \cdots e_d^{l_d} \) are linear combinations of two non-intersecting subsets of basis elements of \( V^\otimes n \).

Likewise, vectors in \( \mathcal{B}_2 \) are linearly independent. Hence \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) are bases for \( S^n V \) and \( \bigwedge^n V \) respectively.

The cardinality of \( \mathcal{B}_1 \) is \( \binom{n+d-1}{n} \) = number of partitions of \( d \) into a sum of \( n \) non-negative integers. The cardinality of \( \mathcal{B}_2 \) is clearly \( \binom{d}{n} \).

(ii) Recall that if \( G \) acts linearly on a vector space \( V \), then the natural action on \( V^\otimes n \) is given by

\[
g(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}) = ge_{i_1} \otimes ge_{i_2} \otimes \cdots \otimes ge_{i_n}.
\]

This action restricts to the \( G \)-invariant subspaces \( S^n V \) and \( \bigwedge^n V \) as follows:

\[
g(e_{i_1} \cdots e_{i_n}) = \frac{1}{n!} \sum_{\sigma \in S_n} ge_{\sigma(i_1)} \otimes \cdots \otimes ge_{\sigma(i_n)},
\]

\[
g(e_{i_1} \wedge \cdots \wedge e_{i_n}) = \frac{1}{n!} \sum_{\sigma \in S_n} \sgn(\sigma) ge_{\sigma(i_1)} \otimes \cdots \otimes ge_{\sigma(i_n)}.
\]

\footnote{The \( 1/n! \) term is optional. It is there so that \( S \) and \( A \) are projections onto \( S^n V \) and \( \bigwedge^n V \) respectively; also, \( S \) may then be interpreted as the ‘average’ of \( S_n \), i.e. \( \frac{1}{|S_n|} \sum_{\sigma \in S} \sigma \).}
When \( g e_i = \lambda_i e_i \) the expressions simplify as follows:

\[
g(e_1, \cdots, e_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \lambda_{\sigma(1)} \cdots \lambda_{\sigma(n)} e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(n)}
\]

\[
= \lambda_1 \cdots \lambda_n e_1 \cdots e_n,
\]

\[
g(e_1 \wedge \cdots \wedge e_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \text{sgn}(\sigma) \lambda_{\sigma(1)} \cdots \lambda_{\sigma(n)} e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(n)}
\]

\[
= \lambda_1 \cdots \lambda_n e_1 \wedge \cdots \wedge e_n,
\]

since \( \lambda_{\sigma(1)} \cdots \lambda_{\sigma(n)} = \lambda_1 \cdots \lambda_n \) for all \( \sigma \in S_n \). Hence, the set of eigenvalues of \( g \) on \( S^n V \) is

\[
\{\lambda_1 \cdots \lambda_n \mid 1 \leq i_1 \leq \cdots \leq i_n \leq d\} = \{\lambda_1^{k_1} \cdots \lambda_d^{k_d} \mid k_1 + \cdots + k_d = n\}
\]

and the set of eigenvalues of \( g \) on \( \Lambda^n V \) is

\[
\{\lambda_1 \cdots \lambda_n \mid 1 \leq i_1 < \cdots < i_n \leq d\}.
\]

(iii) Let \( g \in G \). By (ii),

\[
\text{tr}(g, \Lambda^n V) = \sum_{1 \leq i_1 < \cdots < i_n \leq d} \lambda_1 \cdots \lambda_n
\]

and we have

\[
f(x) = (\lambda_1 - x) \cdots (\lambda_d - x) = (-x)^d + \sum_{n=1}^d \text{tr}(g, \Lambda^n V)(-x)^{d-n}.
\]

So

\[
\text{tr}(g, \Lambda^n V) = (-1)^{d-n} \times \text{coefficient of } x^{d-n} \text{ in } f(x)
\]

for \( n = 1, \ldots, d \).

(iv) Let \( g \in G \). By (ii),

\[
\text{tr}(g, S^n V) = \sum_{k_1 + \cdots + k_d = n} \lambda_1^{k_1} \cdots \lambda_d^{k_d}
\]

and we have

\[
(1 + \lambda_1 x + \lambda_1^2 x^2 + \cdots) \cdots (1 + \lambda_d x + \lambda_d^2 x^2 + \cdots) = 1 + \sum_{n=1}^\infty \text{tr}(g, S^n V)x^n.
\]

Noting that \( (1 - \lambda_i x)^{-1} = 1 + \lambda_i x + \lambda_i^2 x^2 + \cdots \), we have

\[
1 + \sum_{n=1}^\infty \text{tr}(g, S^n V)x^n = \frac{1}{(1 - \lambda_1 x) \cdots (1 - \lambda_d x)}
\]

\[
= \frac{1}{(-x)^d(\lambda_1 - 1/x) \cdots (\lambda_d - 1/x)}
\]

\[
= \frac{1}{(-x)^d f(1/x)}.
\]

So

\[
\text{tr}(g, S^n V) = (-1)^d \times \text{coefficient of } x^{d+n} \text{ in power series expansion of } \frac{1}{f(1/x)}
\]

for \( n = 1, 2, \ldots \).

\[\square\]

**Remark.** Note that the two boxed formulas would have to be changed accordingly if you defined the characteristic polynomial as \( f(x) = \det(xI - g) \).

4. Let \( G \) act on a finite set \( X \), and \( \mathbb{C}[X] \) be the permutation representation.
### 5. Dihedral Group Character Table

Determine the character table of the dihedral group $D_{2n}$.

**Solution.** Straightforward but tedious.

### 6. Dihedral Group Character Table

Determine the character table of the dihedral group $D_{2n}$ of symmetries of the $n$-gon by inducing up representations of the rotation group $\mathbb{Z}/n\mathbb{Z} \hookrightarrow D_{2n}$.

**Solution.** Straightforward.
7. Let $G$ be the Heisenberg group of order $p^3$. This is the subgroup

$$G = \left\{ \begin{pmatrix} 1 & a & x \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, x \in \mathbb{F}_p \right\}$$

of matrices over the finite field $\mathbb{F}_p$, $p$ a prime.

Let $H$ be the subgroup of $G$ of matrices with $a = 0$ and $Z$ be the subgroup of $G$ of matrices with $a = b = 0$.

(i) Show $Z$ is the center of $G$, and that $G/Z = \mathbb{F}_p^2$. Note that this implies that the commutator subgroup $[G, G]$ is contained in $Z$. You can check by explicit computation that it equals $Z$, or if you are lazy you can deduce this from the list of irreducible representations, below.

(ii) By Example Sheet 2, Problem 7, every one-dimensional representation of $G$ is lifted from a representation of $G/Z = \mathbb{F}_p^2$. By Example Sheet 1, Problem 9, the representations of $\mathbb{F}_p^2$ are precisely

$$\chi_{l,m} : \mathbb{F}_p \times \mathbb{F}_p \to \mathbb{C}^\times, \quad (a,b) \mapsto e^{2((a+mb)\pi i)/p}$$

for $l, m = 0, \ldots, p-1$, (ie. we let $\lambda = e^{2\pi i/p}$ and $\mu = e^{2\pi i/p}$ in Example Sheet 1, Problem 9). So the lifted one-dimensional representations of $G$ are

$$\widetilde{\chi}_{l,m} : G \to \mathbb{C}^\times, \quad \begin{pmatrix} 1 & a & x \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mapsto e^{2((a+mb)\pi i)/p}$$

for $l, m = 0, \ldots, p-1$.

(iii) Let

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \zeta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \alpha^a \beta^n \zeta^x.$$

In particular $Z = \{ \zeta^x \mid x = 0, \ldots, p-1 \}$ and $H = \{ \beta^n \zeta^x \mid b, x = 0, \ldots, p-1 \}$.

Recall that $\psi$ has the form $\psi_n : \mathbb{F}_p \to \mathbb{C}^\times$, $x \mapsto e^{2\pi i nx/p}$ for $n = 0, \ldots, p-1$. We will now compute the character of $\text{Ind}_H^G \rho_\psi$.

Case: $a = b = 0$. Let $x \in \{0, \ldots, p-1\}$. Then $\zeta^x \in Z$. So

$$\chi_{\text{Ind}_H^G \rho_\psi}(\zeta^x) = px \rho_\psi(\zeta^x) = p \psi(\zeta^x).$$
Case: \( a = 0, b \neq 0 \). Let \( b \in \{1, \ldots, p-1\} \). Suppose \( \beta^b \) is conjugate to some \( g \in G \), then \( \beta^b Z \) is conjugate to \( gZ \) in the abelian group \( G/Z \), so \( \beta^b Z = gZ \), and therefore \( g = \beta^b \zeta^x \) for some \( x \in \{0, \ldots, p-1\} \). Since \( \beta^b \notin Z \), the conjugacy class \((\beta^b)^G\) does not have size 1, and hence
\[
(\beta^b)^G = \{ \beta^b \zeta^x \mid x = 0, \ldots, p-1 \}.
\]
By the formula for character of induced representations, we have
\[
\chi_{\text{Ind}_{H}^{G} \rho_{\psi}}(\beta^b \zeta^x) = \chi_{\rho_{\psi}}(\beta^b) + \chi_{\rho_{\psi}}(\beta^b \zeta) + \cdots + \chi_{\rho_{\psi}}(\beta^b \zeta^{p-1})
\]
\[
= \psi(\beta^b) + \psi(\beta^b \zeta) + \cdots + \psi(\beta^b \zeta^{p-1})
\]
\[
= \psi(\beta^b) \sum_{i=0}^{p-1}(\psi(\zeta))^i = 0
\]
since \( \psi(\zeta) \) is a pth root of unity.

Case: \( a \neq 0 \). In this case, \( \alpha^a \beta^b \zeta^x \notin H \) and \( (\alpha^a \beta^b \zeta^x)^G \cap H = \emptyset \), so
\[
\chi_{\text{Ind}_{H}^{G} \rho_{\psi}}(\alpha^a \beta^b \zeta^x) = 0
\]
by the definition of induced characters.

Summarizing, if \( \psi = \psi_n \), then
\[
\chi_{\text{Ind}_{H}^{G} \rho_{\psi}}(\alpha^a \beta^b \zeta^x) = \begin{cases} p e^{2\pi i n/p} & \text{if } a = b = 0, \\ 0 & \text{otherwise}, \end{cases}
\]
for \( n = 0, \ldots, p-1 \). \( \text{Ind}_{H}^{G} \rho_{\psi} \) is irreducible since
\[
\langle \chi_{\text{Ind}_{H}^{G} \rho_{\psi}}, \chi_{\text{Ind}_{H}^{G} \rho_{\psi}} \rangle = \frac{1}{|G|} \sum_{g \in G} \left| \chi_{\text{Ind}_{H}^{G} \rho_{\psi}}(g) \right|^2 = \frac{1}{p^3} \sum_{g \in G} \left| \chi_{\text{Ind}_{H}^{G} \rho_{\psi}}(g) \right|^2 = \frac{1}{p^3} \sum_{g \in G} p^2 = 1.
\]
(iv) The \( p^2 \) representations of dimension 1 in (ii), \( \{ \chi_{l,m} \mid l, m = 0, \ldots, p-1 \} \), and the \( p-1 \) representations of dimension \( p \) in (iii), \( \{ \text{Ind}_{H}^{G} \rho_{\psi_n} \mid n = 1, \ldots, p-1 \} \), are all distinct, and the sum of the squares of their degrees is
\[
p^2 \times 1^2 + (p-1) \times p^2 = p^3 = |G|
\]
and so these complete the list of all irreducible representations.

(v) See (iii).

\( \square \)

8. Suppose that \( V \) is a representation of \( G \), and the character \( \chi \) of \( V \) satisfies \( \chi(g) \in \mathbb{R} \) for all \( g \in G \). Show that the trivial representation always occurs in \( V \otimes V \). Conclude that if \( \dim V > 2 \), \( V \otimes^2 \) is the sum of at least three irreducible representations.

**Solution.** Since \( \chi(1)^2 \geq 0 \),
\[
(1, \chi^2) = \frac{1}{|G|} \sum_{g \in G} \chi(g)^2 \geq 0;
\]
it cannot be 0 as \( \chi(1) \neq 0 \) and so the multiplicity of \( 1 \) in \( \chi^2 \) must be non-zero.

If \( \dim V \geq 3 \), then \( \dim \otimes^2 V \geq 6 \) and \( \dim \wedge^2 V \geq 3 \). We know that the trivial representation must occur in either \( \otimes^2 V \) or \( \wedge^2 V \) (or both). Since the trivial representation has dimension 1, either \( \otimes^2 V \) or \( \wedge^2 V \) (or both) must decompose further into two or more irreducible representations. In all cases, we would have at least three irreducible representations.

\( \square \)

9. Let \( (\rho, V) \) be an irreducible representation of a finite group \( G \). How unique is the positive definite hermitian form on \( V \)?
10. Show that the number of irreducible characters of $G$ taking only real values equals the number of
conjugacy classes of elements $g$ such that $g^{-1}$ is conjugate to $g$.

**Solution.** The character table of $G$ may be viewed as an $n \times n$ matrix $A \in \text{GL}(n, \mathbb{C})$ where $n$ is the
number of conjugacy classes of $G$ (A is invertible by row or column orthogonality).

Observe that $\chi$ is an irreducible character if and only if $\chi$ is. So $A = (\chi(g))$ is simply a permutation
of the rows of $A = (\chi(g))$. If we let $P$ denote the permutation matrix, then

$$PA = \overline{A}. \quad (10.1)$$

Since $\chi = \chi$ precisely when $\chi(g) \in \mathbb{R}$ for all $g \in G$, the number of $\mathbb{R}$-valued characters equals the
number of rows in $A$ that are fixed under $A \mapsto \overline{A}$ which in turn equals the trace of $P$.

Now observe that $\chi(g^{-1}) = \chi(g)$, so we may also view $\overline{A} = (\chi(g^{-1}))$ as a permutation of the columns
of $A = (\chi(g))$. Let $Q$ denote the permutation matrix, then

$$AQ = \overline{A}. \quad (10.2)$$

Since $g^G = (g^{-1})^G$ precisely when the column corresponding to $g^G$ equals that corresponding to $(g^{-1})^G$ (distinct conjugacy classes must assume different values on at least one irreducible character
by column orthogonality), the number of conjugacy classes satisfying $g^G = (g^{-1})^G$ equals the number
of columns that are fixed under $A \mapsto \overline{A}$ which in turn equals the trace of $Q$.

By (10.1) and (10.2), $AP = QA$, and the required result follows from

$$\text{tr}(P) = \text{tr}(APA^{-1}) = \text{tr}(Q).$$

\[ \square \]

11. * Let $\rho$ be a faithful irreducible representation of $G$, with character $\chi$, and suppose that $\chi$ takes $t$
distinct values. Show that every irreducible representation of $G$ occurs with non-zero multiplicity in
at least one of

$$1, \rho, \rho \otimes \chi^2, \ldots, \rho \otimes \chi^{t-1}.$$

Hint: Let $\mu$ be an irreducible character, and suppose that $(\chi^j, \mu) = 0$ for $j = 0, \ldots, t-1$. Consider
this an equation

$$(t \times t\text{-matrix of character of powers of } \rho)(\text{vector of } \mu) = 0$$

and invert the matrix to get a contradiction.

**Solution.** Let $\alpha_0, \ldots, \alpha_{t-1} \in \mathbb{C}$ be the $t$ distinct values in $\text{im}(\chi)$ and let $G_i := \chi^{-1}(\alpha_j) = \{x \in G \mid \chi(x) = \alpha_j\}$ for $i = 0, \ldots, t-1$. We may assume $\alpha_0 = \chi(1)$. Since $\chi$ is faithful, $G_0 = \ker(\chi) = \{1\}$.

If $\mu$ is an irreducible character satisfying $(\chi^j, \mu) = 0$ for $j = 0, \ldots, t-1$, then

$$\sum_{i=0}^{t-1} \alpha_i^j \sum_{x \in G_i} \mu(x) = |G| \cdot (\chi^j, \mu) = 0$$

for $j = 0, \ldots, t-1$. Let $\beta_i = \sum_{x \in G_i} \mu(x)$ (note that $\beta_0 = \mu(1)$). Rewriting the system of linear
equations in matrix form, we get

$$\begin{bmatrix}
1 & 1 & \cdots & 1 \\
\alpha_0 & \alpha_1 & \cdots & \alpha_{t-1} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_0^{t-1} & \alpha_1^{t-1} & \cdots & \alpha_{t-1}^{t-1}
\end{bmatrix}
\begin{bmatrix}
\beta_0 \\
\beta_1 \\
\vdots \\
\beta_{t-1}
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
\vdots \\
0
\end{bmatrix}.$$

Since the $\alpha_i$’s are all distinct and the Vandermonde matrix is invertible, we have $\beta_i = 0$ for all
$i = 0, \ldots, t-1$. In particular, $0 = \beta_0 = \mu(1) \neq 0$ gives the required contradiction. \[ \square \]

**Remark.** The rows $v_0, \ldots, v_{t-1}$ of the Vandermonde matrix are linearly independent since a linear
combination

$$a_0 v_0 + \cdots + a_{t-1} v_{t-1} = 0$$

gives a polynomial $a_0 + a_1 X + \cdots + a_{t-1} X^{t-1}$ that has $t$ distinct zeroes $1, \alpha_0, \ldots, \alpha_{t-1}$ and so must
be the zero polynomial, i.e. $a_0 = \cdots = a_{t-1} = 0$. 

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12. * Prove the following theorem of Burnside. If \( \rho \) is an irreducible representation of \( G \) with character \( \chi \) and \( \dim \rho > 1 \) then there exists a \( g \in G \) such that \( \chi(g) = 0 \).

**SOLUTION.** (Sketch) Let \( \psi \) be a character (not necessarily irreducible) of the cyclic group \( \mathbb{C} \cong \mathbb{Z}/n\mathbb{Z} \). Let \( X = \{ g \in \mathbb{C} \mid \langle g \rangle = \mathbb{C} \} \). We could show that \( \prod_{g \in X} \psi(g) \) is an integer. Hence, if \( \psi(g) \neq 0 \), then \( \prod_{g \in X} |\psi(g)|^2 \geq 1 \) and so \( \sum_{g \in X} |\psi(g)|^2 \geq |X| \) (by the fact that Arithmetic Mean \( \geq \) Geometric Mean).

Now define an equivalence relation on the elements of \( G \) by \( g \sim h \) iff \( \langle g \rangle = \langle h \rangle \). Let \( X_i = \{ g \in G \mid \langle g \rangle = \langle g_i \rangle \}, i = 1, \ldots, m \) be the equivalence classes. If \( \chi(g) = \chi(g') \neq 0 \) for all \( g \in G \), then \( \chi_{\text{Res}}(\rho)(g) \neq 0 \) for all \( g \in G \) and we may apply the above result to \( X_i \) and \( \psi = \chi_{\text{Res}}(\rho) \) to get

\[
\sum_{g \in X_i} |\psi(g)|^2 \geq |X_i|.
\]

Let \( g_1 = 1 \). Then \( X_1 = \{1\} \). So \( \sum_{g \in X_1} |\psi(g)|^2 = |\psi(1)|^2 > 1 = |X_1| \) since \( \dim \rho > 1 \).

From \( (\psi, \psi) = 1 \), we get

\[
|G| = \sum_{g \in G} |\psi(g)|^2 = \sum_{i=1}^m \sum_{g \in X_i} |\psi(g)|^2 > \sum_{i=1}^m |X_i| = |G|,
\]

which gives a contradiction.

Hence \( \chi(g) = 0 \) for some \( g \in G \).

\( \square \)

13. Show that an abelian group has a faithful irreducible representation if and only if it is cyclic.

**SOLUTION.** We assume that the group \( G \) is finite and the representation \( \rho \) is over \( \mathbb{C} \). Recall that complex irreducible representations of abelian groups are one-dimensional (Example Sheet 1, Problem 9) and that each \( \rho(x) \in \mathbb{C}^\times \) is an \( N \)th root of unity where \( N = |G| \) (Example Sheet 1, Problem 7). So \( \rho(x) \in \partial \mathbb{D} = \{ z \in \mathbb{C} \mid |z| = 1 \} \) for all \( x \in G \).

If \( G \) is abelian and \( \rho : G \to \mathbb{C}^\times \) is faithful, then \( G \cong \rho(G) \). But \( \rho(G) \leq \partial \mathbb{D} \cong S^1 \) and since the only finite subgroups of \( S^1 \) are either dihedral or cyclic, we must have \( G \) is cyclic as it is abelian.

If \( G = \langle x \rangle \) is a cyclic group, then \( \rho : G \to \mathbb{C}^\times, x^n \mapsto e^{2\pi i n/N} \) is clearly a faithful representation.

\( \square \)

14. Let \( \Gamma_r = \{ A \subseteq \{1, \ldots, n\} \mid |A| = r \} \), and \( \chi_r \) be the character of the permutation representation \( \mathbb{C}[\Gamma_r] \) of \( \mathbb{S}_n \). Compute \( (\chi_r, \chi_s) \) and hence decompose \( \mathbb{C}[\Gamma_r] \) as a representation of \( \mathbb{S}_n \).

**SOLUTION.** (Impetus for this solution provided by Dominic Pinto, Caius College) By Problem 9 in Example Sheet 2,

\[
(\chi_r, \chi_s) = \#\mathbb{S}_n\text{-orbits on } \Gamma_r \times \Gamma_s.
\]

Let \( \Delta := \{(x, x) \mid x \in X\} \). Recall that \( \mathbb{S}_n \) acts transitively on the \( \mathbb{S}_n \)-stable subsets \( \Delta \) and \( X \times X \setminus \Delta \). It follows that the \( \mathbb{S}_n \)-orbits on \( \Gamma_r \times \Gamma_s \) are

\[
O_i = \{ A \times B \in \Gamma_r \times \Gamma_s \mid |A \cap B| = i \}.
\]

for \( i = \max(0, s + r - n), \ldots, \min(s, r) \) (a moment’s thought would show that these are the only possible values \( i \) could take).

Hence

\[
(\chi_r, \chi_s) = \min(s, r) - \max(0, s + r - n) + 1.
\]

Note that (14.1) also follows from a direct argument: if \( A \times B, C \times D \in \Gamma_r \times \Gamma_s \) and \( |A \cap B| = i = |C \cap D| \), let \( A \cap B = \{x_1, \ldots, x_k\}, A \setminus B = \{x_{k+1}, \ldots, x_r\}, B \setminus A = \{x'_1, \ldots, x'_s\}, X \setminus (A \cup B) = \{x''_{r+1}, \ldots, x''_{s+k}\} \), \( C \cap D = \{y_1, \ldots, y_s\}, C \setminus D = \{y'_{s+1}, \ldots, y'_{r+s}\}, X \setminus (C \cup D) = \{y''_{r+s+1}, \ldots, y''_{r+s+k}\} \).

Then the permutation in \( \mathbb{S}_n \) defined by

\[
\begin{pmatrix}
  x_1 & \cdots & x_k & x_{k+1} & \cdots & x_r & x'_{s+1} & \cdots & x'_s & x''_{r+1} & \cdots & x''_{s+k} \\
  y_1 & \cdots & y_s & y_{s+1} & \cdots & y_{r+s} & y'_{r+s+1} & \cdots & y'_{r+s+k} & y''_{r+s+1} & \cdots & y''_{r+s+k}
\end{pmatrix}
\]

would send \( A \times B \) to \( C \times D \), i.e. they lie in the same \( \mathbb{S}_n \)-orbit. Since \( \mathbb{S}_n \) permutes elements in \( \Delta \), \( A \times B \) and \( C \times D \) with \( |A \cap B| \neq |C \cap D| \) cannot lie in the same \( \mathbb{S}_n \)-orbit.

To decompose \( \mathbb{C}[\Gamma_r] \) into irreducibles, observe that
\( \Gamma_1 = \{1, 2, \ldots, n\} \), so \( \mathbb{C}[\Gamma_1] \) is just the usual permutation representation of \( S_n \) which we know breaks up into exactly two representations: the trivial representation \( \mathbf{1} \) and an \( (n-1) \)-dimensional irreducible representation \( V_1 \) (see your lecture notes).

2. \( \Gamma_n \) is a singleton, so \( \mathbb{C}[\Gamma_n] \) is the trivial representation. Hence \( \chi_n = \mathbf{1} \).

3. \( \mathbb{C}[\Gamma_r] \) and \( \mathbb{C}[\Gamma_{n-r}] \) are equivalent representations of \( S_n \). The obvious \( G \)-isomorphism being \( \mathbb{C}[\Gamma_r] \to \mathbb{C}[\Gamma_{n-r}], A \mapsto X \setminus A \) for \( A \in \Gamma_r \). \( G \)-equivariance follows from \( \sigma(X \setminus A) = X \setminus \sigma A \) for \( \sigma \in S_n \). Hence \( \chi_r = \chi_{n-r} \).

By 2, (14.2) with \( s = n \) gives 
\[
(\chi_r, \mathbf{1}) = (\chi_r, \chi_n) = 1.
\]
By 3, we may assume that \( s \leq r \leq n/2 \), so \( s + r \leq n \) and (14.2) gives 
\[
(\chi_r, \chi_s) = s + 1.
\]

For \( r = 2 \), \( (\chi_2, \mathbf{1}) = 1 \) and \( (\chi_2, \chi_1) = 2 \) imply \( \mathbf{1} \) and \( V_1 \) both occur with multiplicity 1 in \( \mathbb{C}[\Gamma_2] \). Since \( (\chi_2, \chi_2) = 3 \), there is a third irreducible representation \( V_2 \) of dimension \( \deg(\chi_2) - \deg(\chi_1) = \binom{n}{2} - \binom{n}{1} \).

In general, since \( (\chi_r, \chi_{r-1}) = r \) and \( (\chi_r, \chi_r) = r + 1 \), \( \mathbb{C}[\Gamma_r] \) contains an irreducible representation \( V_r \) of degree \( \deg(\chi_r) - \deg(\chi_{r-1}) = \binom{n}{r} - \binom{n}{r-1} \) that is not in \( \mathbb{C}[\Gamma_{r-1}] \). Inductively we get 
\[
\mathbb{C}[\Gamma_r] = \mathbf{1} \oplus V_1 \oplus \cdots \oplus V_r.
\]

15. Using Problems 14 and 4, decompose \( S^m \mathbb{C}^n \) as representation of \( S_n \) for all \( m \geq 1 \).

**Solution.** Let \( X = \{1, 2, \ldots, n\} \). Note that \( \mathbb{C}[X] \cong \mathbb{C}^n \) as \( S_n \)-modules.

\[
S^m \mathbb{C}^n = S^m \mathbb{C}[X] = C[S^m X] = C[M_m] \oplus C[M_{m-1}] \oplus \cdots \oplus C[\Gamma_1]
\]
\[
= (\mathbf{1} \oplus V_1 \oplus \cdots \oplus V_m) \oplus (\mathbf{1} \oplus V_1 \oplus \cdots \oplus V_{m-1}) \oplus \cdots \oplus (\mathbf{1} \oplus V_1)
\]
\[
= m \mathbf{1} \oplus (m-1)V_1 \oplus \cdots \oplus V_n
\]

16. Let \( V \) be an irreducible representation of \( G \), with character \( \chi \). Show that \( |G|^{-1} \sum_{g \in G} \chi(g^2) \) is either 0, 1 or \(-1\), and that this zero precisely when \( \chi \) is not \emph{real} valued, ie. when \( V \) is not isomorphic to \( V^* \). (Hint: consider \( V \otimes V = S^2 V \oplus \Lambda^2 V \).

**Solution.** Since \( \chi \) is irreducible, it occurs with multiplicity one in itself and if it is not real valued, the \( \overline{\chi} \) is a different irreducible character, ie.

\[
\langle \chi, \overline{\chi} \rangle = \begin{cases} 1 \quad \text{if } \chi \text{ is real valued,} \\ 0 \quad \text{if } \chi \text{ is not real valued.} \end{cases}
\]

Since
\[
\langle \chi, \overline{\chi} \rangle = \frac{1}{|G|} \sum_{g \in G} \chi^2(g) = \langle \chi^2, \mathbf{1} \rangle = \langle \chi_{S^2 V} + \chi_{\Lambda^2 V}, \mathbf{1} \rangle = \langle \chi_{S^2 V}, \mathbf{1} \rangle + \langle \chi_{\Lambda^2 V}, \mathbf{1} \rangle,
\]
we have the following possibilities

1. \( \langle \chi_{S^2 V}, \mathbf{1} \rangle = 1 \) and \( \langle \chi_{\Lambda^2 V}, \mathbf{1} \rangle = 0 \) when \( \chi \) is real valued,
2. \( \langle \chi_{S^2 V}, \mathbf{1} \rangle = 0 \) and \( \langle \chi_{\Lambda^2 V}, \mathbf{1} \rangle = 0 \) when \( \chi \) is not real valued,
3. \( \langle \chi_{S^2 V}, \mathbf{1} \rangle = 0 \) and \( \langle \chi_{\Lambda^2 V}, \mathbf{1} \rangle = 1 \) when \( \chi \) is real valued.

Hence this gives
\[
\frac{1}{|G|} \sum_{g \in G} \chi(g^2) = \langle \chi_{S^2 V}, \mathbf{1} \rangle - \langle \chi_{\Lambda^2 V}, \mathbf{1} \rangle = \begin{cases} 1 \quad \text{if } \circ \text{ holds,} \\ 0 \quad \text{if } \circ \text{ holds,} \\ -1 \quad \text{if } \circ \text{ holds.} \end{cases}
\]

Note that \( \circ \) happens precisely when \( \chi \) is not real valued.
17. Let $G \leq \text{GL}_n(\mathbb{C})$, $G$ a finite group. Show that $\sum_{g \in G} \text{tr}(g) = 0$ implies $\sum_{g \in G} g = 0$.

Solution. We may regard $G \hookrightarrow \text{GL}_n(\mathbb{C})$, $g \mapsto g$ as a representation of $G$. Recall that every representation of a finite group is equivalent to a unitary representation (the proof is similar to that in Example Sheet 1, Problem 8(i)). In this case, it means that we may assume that $G \leq \text{U}(n)$, i.e. $g^* := g^T = g^{-1}$ for all $g \in G$. Let $A = \sum_{g \in G} g \in \mathbb{M}_n(\mathbb{C})$. Observe that

① $A$ is self-adjoint (a.k.a. hermitian) since

$$A^* = \sum_{g \in G} g^* = \sum_{g \in G} g^{-1} = \sum_{g \in G} g = A.$$ 

② For all $x \in G$,

$$xA = \sum_{g \in G} xg = \sum_{g \in G} g = A.$$ 

Suppose $A \neq 0$. Then by ①, $A^2 = AA^*$ is positive definite and so has eigenvalues $\lambda_1, \ldots, \lambda_n \in (0, \infty)$. Since

$$A^2 = \left( \sum_{g \in G} g \right) A = \sum_{g \in G} gA = \sum_{g \in G} A,$$

we get $\text{tr}(A^2) = |G| \text{tr}(A)$ and $\text{tr}(A^2) = \lambda_1 + \cdots + \lambda_n > 0$ would then imply that $\text{tr}(A) > 0$. □

Remark. Given any $G \leq \text{GL}_n(\mathbb{C})$, $G \hookrightarrow \text{GL}_n(\mathbb{C})$ is often known as the natural representation of $G$.