Solutions to Example Sheet 2

All groups are finite; all representations are over complex vector spaces.

Group G acts on set X ∘ **orbit** of $x \in X$ is $x^G := \{gx \in X \mid g \in G\}$ ∘ **stabilizer** of $x \in X$ is $G_x := \{g \in G \mid gx = x\}$ ∘ **fixed point set** of $g \in G$ is $X^g := \{x \in X \mid gx = x\}$ Special case: G acts on itself via conjugation ∘ **conjugacy class** of $x \in G$ is $x^G := \{gxg^{-1} \in G \mid g \in G\}$ ∘ **centralizer** of $x \in G$ is $C_G(x) := G_x = G^x = \{g \in G \mid gx = xg\}$ ∘ **center** of G is $Z(G) := \{g \in G \mid gx = xg \text{ for all } x \in G\} = \bigcap_{x \in G} C_G(x)$ ∘ **commutator** of x and y is $[x, y] := x^{-1}y^{-1}xy$ ∘ **commutator subgroup** of G is $G' := \langle [x, y] \mid x, y \in G\rangle$ Recall: $\lvert x^G \rvert = [G : G_x]$

1. (a) Let ρ be a representation of G with character χ, and suppose dimension of ρ is d. Show $\ker \rho = \{g \in G \mid \chi(g) = d\}$.

(b) Show $|\chi(g)| \leq d$ for all $g \in G$, and that if $|g| = d$, then $\rho(g) = \lambda I_d$, where λ is a root of unity.

Solution. As in Problem **7**, Example Sheet **1** (applied to the cyclic subgroup $\langle g \rangle \trianglelefteq G$), we may pick a basis so that $\rho(g) = \operatorname{diag}(\lambda_1, \dots, \lambda_d)$ where the λ_i's are nth roots of unity, $n = \operatorname{ord}_G(g) = |\langle g \rangle|$. Note that $\chi(g)$ is basis independent and so $\chi(g) = \lambda_1 + \cdots + \lambda_d$.

(a) Clearly if $g \in \ker \rho$, then $\chi(g) = \operatorname{tr} \rho(g) = \operatorname{tr}(I_d) = d$. Conversely, $d = \chi(g) = \lambda_1 + \cdots + \lambda_d$ could only happen if $\lambda_i = 1$ for all $i = 1, \dots, d$. Hence $\rho(g) = I_d$.

(b) By the triangle inequality $|\chi(g)| = |\lambda_1 + \cdots + \lambda_d| \leq |\lambda_1| + \cdots + |\lambda_d| = d$. Equality occurs iff $\arg(\lambda_1) = \cdots = \arg(\lambda_d) = \theta$ for some θ. So $\lambda_1 = \cdots = \lambda_d = e^{i\theta}$ which must be a root of unity since all λ_i's are and so we have $\rho(g) = e^{i\theta} I_d$. ☐

2. Let χ be a character of G, and suppose $g \in G$ has order 2, ie. $g^2 = 1$. Show $\chi(g) \in \mathbb{Z}$, and $\chi(g) \equiv \chi(1) \pmod 2$.

Solution. Let d be the dimension of χ. As in the solution of Problem **1**, $\chi(g)$ is a sum of d square roots of unity, ie. ± 1. Since $-1 \equiv 1 \pmod 2$, $\chi(g) \equiv d = \chi(1) \pmod 2$. ☐

3. (a) Let $G = \{\pm 1, \pm i, \pm j, \pm k\}$ be the "quaternion group" $\mathbf{Q}_8$, ($ij = k = -ji$, $i^2 = j^2 = k^2 = -1$). Regarding G as a subgroup of the quaternions $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$, we get a two-dimensional representation of G. Show this is irreducible.

(b) Conclude the remaining irreducible representations of G are one-dimensional. Find them. Write the character table of G.

(c) Let $G = \mathbf{D}_8$ = symmetries of the square. Show G acts on $\mathbb{C}^2$ irreducibly, and determine the character table of G.

(d) Compare the character tables of $\mathbf{D}_8$ and $\mathbf{Q}_8$. Comment?

Solution[1]. First, a few easy facts about $\mathbf{Q}_8$. Clearly ± 1 are in $Z(\mathbf{Q}_8)$; $jij^{-1} = kik^{-1} = -i$ and so $i^{\mathbf{Q}_8} = \{\pm i\}$. By the symmetry in the relations on i, j, k, we must also have $j^{\mathbf{Q}_8} = \{\pm j\}$ and $k^{\mathbf{Q}_8} = \{\pm k\}$ and thus $Z(\mathbf{Q}_8) = \{\pm 1\}$. Since $[i, j] = i^{-1}j^{-1}ij = ijij = -i^2j^2 = -1$, we have $[a, b] = -1$ for every $a, b \in \{\pm i, \pm j, \pm k\}$, $a \neq b$, again by symmetry and $\pm 1 \in Z(\mathbf{Q}_8)$. So $\mathbf{Q}_8' = \{\pm 1\}$ and $\mathbf{Q}_8/\mathbf{Q}_8' = \{\overline{1}, \overline{i}, \overline{j}, \overline{ij}\} \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. The 'header' of the character table is:

[1]The solution to this problem is worked out in full details. You are not required to write most of these steps down in your solutions (and please don't). The obvious steps will be dropped in future problems.

$x^{\mathbf{Q}_8}$	1	-1	$\{\pm i\}$	$\{\pm j\}$	$\{\pm k\}$
$\lvert C_{\mathbf{Q}_8}(x)\rvert$	8	8	4	4	4

(a) If we regard $\mathbb{H}$ as a *left* $\mathbb{C}$-vector space (ie. we write the scalars on the left), then *right* multiplication by elements in $\mathbf{Q}_8$ gives rise to a $\mathbf{Q}_8$-action on $\mathbb{H}$, (ie. $\mathbf{Q}_8 \times \mathbb{H} \to \mathbb{H}$, $(g, \alpha) \mapsto \alpha g$). Pick the basis $\{1, j\}$ and determine the $\mathbf{Q}_8$ action (to make things clearer, $\circ$ is used to denote group action and $\cdot$ is used to denote scalar multiplication):

$$\begin{aligned} 1 \circ (\pm i) &= \pm i \cdot 1 + 0 \cdot j & 1 \circ (\pm j) &= 0 \cdot 1 \pm 1 \cdot j & 1 \circ (\pm k) &= 0 \cdot 1 + (\pm i) \cdot j \\ j \circ (\pm i) &= 0 \cdot 1 + (\mp i) \cdot j & j \circ (\pm j) &= \mp 1 \cdot 1 + 0 \cdot j & j \circ (\pm k) &= \pm i \cdot 1 + 0 \cdot j \end{aligned}$$

and this gives a matrix representation $\rho : \mathbf{Q}_8 \to \mathrm{GL}_2(\mathbb{C})$ with

$$\rho(\pm 1) = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, \quad \rho(\pm i) = \begin{pmatrix} \pm i & 0 \\ 0 & \mp i \end{pmatrix}, \quad \rho(\pm j) = \begin{pmatrix} 0 & \pm 1 \\ \mp 1 & 0 \end{pmatrix}, \quad \rho(\pm k) = \begin{pmatrix} 0 & \pm i \\ \pm i & 0 \end{pmatrix}.$$

Computing the trace gives the following entry in the character table:

$x^{\mathbf{Q}_8}$	1	-1	$\{\pm i\}$	$\{\pm j\}$	$\{\pm k\}$
χ_ρ	2	-2	0	0	0

ρ is irreducible since

$$\begin{aligned} \langle \chi_\rho, \chi_\rho \rangle &= \frac{1}{|\mathbf{Q}_8|} \sum_{x \in \mathbf{Q}_8} \chi_\rho(x) \overline{\chi_\rho(x)} \\ &= \sum_{x^{\mathbf{Q}_8}} \frac{1}{|C_{\mathbf{Q}_8}(x)|} \chi_\rho(x) \overline{\chi_\rho(x)} \\ &= \frac{1}{8} \left(\frac{2^2}{8} + \frac{(-2)^2}{8} + \frac{0}{4} + \frac{0}{4} + \frac{0}{4} \right) = 1. \end{aligned}$$

(b) Let d_1, d_2, d_3 be the dimensions of the other non-trivial irreducible characters. Then since

$$1^2 + d_1^2 + d_2^2 + d_3^2 + 2^2 = |\mathbf{Q}_8| = 8,$$

we must have $d_1 = d_2 = d_3 = 1$. Hence the remaining irreducible representations are one-dimensional. By Problem **7**, the linear characters are exactly those lifted from $\mathbf{Q}_8/\mathbf{Q}_8' \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. Recall from the Problem **9**, Example Sheet **1** that the irreducible (thus one-dimensional) representations ($\equiv$ linear characters) of $\mathbb{Z}/2 \times \mathbb{Z}/2$ are:

$\mathbb{Z}/2 \times \mathbb{Z}/2$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$\mathbb{1}$	1	1	1	1
χ_1	1	-1	1	-1
χ_2	1	1	-1	-1
χ_3	1	-1	-1	1

Lifting the characters on $\mathbb{Z}/2 \times \mathbb{Z}/2$ to $\mathbf{Q}_8$ means to see where the map

$$\mathbf{Q}_8 \xrightarrow{\pi} \mathbf{Q}_8/\mathbf{Q}_8' \xrightarrow{\simeq} \mathbb{Z}/2 \times \mathbb{Z}/2 \xrightarrow{\chi} \mathbb{C}^\times$$

sends the conjugacy classes in $\mathbf{Q}_8$. For instance if we look at χ_1, then

$$\begin{aligned} 1 &\mapsto \bar{1} \mapsto (0,0) \mapsto 1 & \pm i &\mapsto \bar{i} \mapsto (1,0) \mapsto -1 & \pm k &\mapsto \overline{ij} \mapsto (1,1) \mapsto -1 \\ -1 &\mapsto \bar{1} \mapsto (0,0) \mapsto 1 & \pm j &\mapsto \bar{1} \mapsto (0,1) \mapsto 1 \end{aligned}$$

and we get (abusing notation, we use χ_1 for the lifted character too):

$x^{\mathbf{Q}_8}$	1	-1	$\{\pm i\}$	$\{\pm j\}$	$\{\pm k\}$
χ_1	1	1	-1	1	-1

Repeating this for χ_2 and χ_3, we obtain the full character table of $\mathbf{Q}_8$:

$x^{\mathbf{Q}_8}$	1	-1	$\{\pm i\}$	$\{\pm j\}$	$\{\pm k\}$
$\mathbb{1}$	1	1	1	1	1
χ_1	1	1	-1	1	-1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	-1	1
χ_ρ	2	-2	0	0	0

(c) See: *Some notes on* $\mathbf{D}_n$, $\mathbf{A}_n$ *and* $\mathbf{S}_n$.

(d) $\mathbf{D}_8 \not\cong \mathbf{Q}_8$ since $\mathbf{D}_8$ has two elements of order 2: a^2 and b, while $\mathbf{Q}_8$ has only one element of order 2: -1. So the character table of a group does not determine the group up to isomorphism. □

4. Determine the character table for $\mathbf{D}_{12}$ = symmetries of the hexagon.

SOLUTION. See: *Some notes on* $\mathbf{D}_n$, $\mathbf{A}_n$ *and* $\mathbf{S}_n$. □

5. Determine the character table for $\mathbf{D}_{10}$ = symmetries on the 5-gon. For every representation V of $\mathbf{D}_{10}$, we may regard it as a representation of $\mathbb{Z}/5 \hookrightarrow \mathbf{D}_{10}$. Determine, for each irreducible representation V of $\mathbf{D}_{10}$, how it decomposes into irreducible characters for $\mathbb{Z}/5$.

SOLUTION. See: *Some notes on* $\mathbf{D}_n$, $\mathbf{A}_n$ *and* $\mathbf{S}_n$. □

6. Determine the character table for the alternating group $\mathbf{A}_4$.

SOLUTION. See: *Some notes on* $\mathbf{D}_n$, $\mathbf{A}_n$ *and* $\mathbf{S}_n$. □

7. Describe the commutator subgroup of a group in terms of the character table of G.

SOLUTION. We claim that the rows in the character table of G with 1 in the first column are precisely the characters lifted from $G' \trianglelefteq G$. If χ, a character of G, is lifted from χ', a character of G/G', then $\chi = \chi' \circ \pi$ and so χ is one-dimensional, ie. $\chi(1) = 1$. Conversely, if $\chi(1) = 1$, then we have to find a $\chi' : G/G' \to \mathbb{C}^\times$ such that $\chi = \chi' \circ \pi$. The obvious choice would of course be

$$G/G' \xrightarrow{\chi'} \mathbb{C}^\times, \quad xG' \mapsto \chi(x).$$

Since our definition of χ' involves a choice of coset representative, we have to show that this is a well-defined map, ie. the image is independent of the coset representative we chose: $\chi(x) = \chi(xg)$ for all $g \in G'$ or equivalently $\chi(g) = 1$ for all $g \in G'$. The last condition is just $G' \leq \ker(\chi')$ and this is obvious since $G/\ker(\chi') \leq \mathbb{C}^\times$ and so is abelian, thus $G' \leq \ker(\chi')$. Observe also the characters lifted from the irreducible character of G/G' are all distinct (clear from how a lifted character is defined) and so the character table of G contains exactly $|G/G'|$ rows of dimension one.

We further claim that

$$G' = \bigcap_{\chi(1)=1} \ker(\chi). \tag{7.1}$$

In the earlier part, we have already shown that $G' \subseteq \ker(\chi)$ for all one-dimensional character χ. So "$\subseteq$" is done. For the converse, recall that:

① By the earlier part, all irreducible characters of G/G' are of the form χ' where $\chi' \circ \pi = \chi$ is a one-dimensional character of G.

② In general, if $x \in G$ satisfies $\chi(x) = 1$ for every irreducible character of G, then $x = 1$.

So we have

$$\begin{aligned} g \in \textstyle\bigcap_{\chi(1)=1} \ker(\chi) &\Longrightarrow \chi(g) = 1 \quad \text{for all one-dimensional character } \chi \text{ of } G \\ &\overset{①}{\Longrightarrow} \chi'(gG') = \chi' \circ \pi(g) = \chi(g) = 1 \quad \text{for all irreducible character } \chi' \text{ of } G/G' \\ &\overset{②}{\Longrightarrow} gG' = G' \\ &\Longrightarrow g \in G' \end{aligned}$$

and we get "$\supseteq$". □

Remarks. (7.1) allows us to get G' from the character table of G. Recall that $\ker(\chi) = \{x \in G \mid \chi(x) = \chi(1)\}$ and so in this case G' is the collection of those $x \in G$ satisfying $\chi(x) = 1$ for every one-dimensional in the character table (see example below).

Example. It is easy to tell which are the elements in G' once you are given the character table of G. In Problem **8**, we will see that the character table for the non-abelian group of order 21, $\mathbf{F}_{7,3} = \langle x, y \mid x^7 = y^3 = 1, y^{-1}xy = x^2 \rangle$ is:

$\mathbf{F}_{7,3}$	1	$x^{\mathbf{F}_{7,3}}$	$(x^3)^{\mathbf{F}_{7,3}}$	$b^{\mathbf{F}_{7,3}}$	$(b^2)^{\mathbf{F}_{7,3}}$
$\lvert g^{\mathbf{F}_{7,3}} \rvert$	1	3	3	7	7
$\mathbb{1}$	1	1	1	1	1
χ_1	1	1	1	ζ	$\overline{\zeta}$
χ_2	1	1	1	$\overline{\zeta}$	ζ
χ_3	3	γ	$\overline{\gamma}$	0	0
χ_4	3	$\overline{\gamma}$	γ	0	0

So $G' = \{1\} \cup x^{\mathbf{F}_{7,3}} \cup (x^3)^{\mathbf{F}_{7,3}} = \{1\} \cup \{x, x^2, x^4\} \cup \{x^3, x^5, x^6\} = \langle x \rangle$, ie. union of the conjugacy classes corresponding to the block of 1's in the table.

8. The table below is part of the character table of a finite group, but some of the rows are missing

$\lvert g^G \rvert$	1	3	3	7	7
χ_a	1	1	1	ζ	$\overline{\zeta}$
χ_b	3	γ	$\overline{\gamma}$	0	0
χ_c	3	γ	$\overline{\gamma}$	0	0

$$\gamma = \tfrac{1}{2}(-1 + \sqrt{7}i)$$
$$\zeta = \tfrac{1}{2}(-1 + \sqrt{3}i)$$

(a) complete the table

(b) describe the group in terms of generators and relations

(a) The table is obviously missing the trivial character $\mathbb{1}$. Since χ_a is irreducible, so must $\overline{\chi}_a$; the latter is a character distinct from $\mathbb{1}$, χ_a, χ_b, χ_c and so must also be on the table. Hence we have

$\lvert g^G \rvert$	1	3	3	7	7
$\mathbb{1}$	1	1	1	1	1
χ_a	1	1	1	ζ	$\overline{\zeta}$
$\overline{\chi}_a$	1	1	1	$\overline{\zeta}$	ζ
χ_b	3	γ	$\overline{\gamma}$	0	0
χ_c	3	$\overline{\gamma}$	γ	0	0

(b) The order of the conjugacy classes tells us that G is non-abelian and has order 21. By Problem **7**, the one-dimensional characters are precisely the ones that are lifted from the characters G/G'. Since there are three of these, $|G/G'| = 3$, so $|G'| = 7$ and so $G' \cong \mathbb{Z}/7$. Let $x \in G$ be a generator of G' and so it has order 7. Let $|y^G| = 7$, then $\chi_a(y) = \zeta$ (or $\overline{\zeta}$) is a third root of unity. Note that χ_a is one-dimensional and so a homomorphism of $G \to \mathbb{C}^\times$. The order of y is either 3, 7 or 21. It cannot be 21 or G would be cyclic (thus abelian); it cannot be 7 since $\chi_a(y^7) = \chi_a(y)^7 = \zeta^7 \neq 1$; so y has order 3. x and y together must generate a group of at least 21 and so we have found the generators of G. It remains to obtain a non-trivial relation between x and y. Since $\langle x \rangle = G' \trianglelefteq G$, $yxy^{-1} = x^r$ for some r with $\gcd(r, 7) = 1$ (since $y^{-1}xy$ must have the same order as x) and we may regard r as an element of $(\mathbb{Z}/7)^*$. Observe that $y^{-1}x^n y = (y^{-1}xy)^n = x^{rn}$ and so

$$x = y^{-3}xy^3 = y^{-2}(y^{-1}xy)y^2 = y^{-2}x^r y^2 = y^{-1}(y^{-1}x^r y)y^{-1} = y^{-1}x^{r^2}y = x^{r^3}.$$

So $r^3 \equiv 1 \pmod 7$, ie. order of r in $(\mathbb{Z}/7)^*$ divides 3; if it were 1, then we get $yx = yx$ and consequently G abelian, so it must be 3. By raising elements in $(\mathbb{Z}/7)^* = \{\pm 1, \pm 2, \pm 3\}$ to third powers, we may find that the elements of order 3 explicitly and so get $r = 2, 4$. Hence we have

$$G = \langle x, y \mid x^7 = y^3 = 1, y^{-1}xy = x^2 \rangle \quad \text{or} \quad \langle x, y \mid x^7 = y^3 = 1, y^{-1}xy = x^4 \rangle.$$

It is easy to see that this two presentations define isomorphic groups (taking $z = y^2$ in the first presentation and rewriting the relations in terms of x and z gives the second presentation). $\square$

Remarks. Note that Part (b) cannot be done in general since the presentation of a group determines it uniquely up to isomorphism while as Problem **3** shows, the character table of a group doesn't determine the group uniquely. Also, the way to arrived at the generators and relations is quite ad hoc — it depends on how much group theory you assume. If you know that there exists exactly one non-abelian group of order pq, $p \equiv 1 \pmod{q}$: $\mathbf{F}_{p,q} = \langle a, b \mid a^p = b^q = 1, b^{-1}ab = a^r \rangle$ where $r \in (\mathbb{Z}/p)^*$ has order q (a standard exercise in the application of Sylow's theorems), then the header of the character table alone would tell you what the group is non-abelian of order 21 and thus must be $\mathbf{F}_{7,3}$.

9. Show $\dim_{\mathbb{C}} \operatorname{Hom}_G(\mathbb{C}[X], \mathbb{C}[Y]) =$ number of G-orbits on $X \times Y$, where X and Y are finite G sets, and $\mathbb{C}[X]$ and $\mathbb{C}[Y]$ are the corresponding permutation representations.

SOLUTION. Let χ_1 and χ_2 be the character afforded on the permutation G-modules $\mathbb{C}[X]$ and $\mathbb{C}[Y]$ respectively. Recall that $\chi_1(g) = |X^g|$ and $\chi_2(g) = |Y^g|$. Since

$$\dim_{\mathbb{C}} \operatorname{Hom}_G(\mathbb{C}[X], \mathbb{C}[Y]) = \langle \chi_1, \chi_2 \rangle,$$

we just need to compute the RHS. To do this, we observe two easy facts:

① $(X \times Y)^g = \{(x, y) \in X \times Y \mid g(x, y) = (gx, gy) = (x, y)\} = X^g \times Y^g$.

② If Z is a G-set, then $\coprod_{g \in G} Z^g = \{(z, g) \in Z \times G \mid z \in Z^g\} = \{(z, g) \in Z \times G \mid gz = z\} = \{(z, g) \in Z \times G \mid g \in G_z\} = \coprod_{z \in Z} G_z$.

Let $Z = X \times Y$. We have

$$\begin{aligned}\langle \chi_1, \chi_2 \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi_1(g)\overline{\chi_2(g)} = \frac{1}{|G|} \sum_{g \in G} |X^g||Y^g| \overset{①}{=} \frac{1}{|G|} \sum_{g \in G} |Z^g| = \frac{1}{|G|} \left| \coprod_{g \in G} Z^g \right| \\ &\overset{②}{=} \frac{1}{|G|} \left| \coprod_{z \in Z} G_z \right| = \frac{1}{|G|} \sum_{z \in Z} |G_z| = \sum_{z \in Z} \frac{|G_z|}{|G|} = \sum_{z \in Z} \frac{1}{|z^G|} \overset{③}{=} \sum_{i=1}^{n} \sum_{z \in z_i^G} \frac{1}{|z^G|} = n\end{aligned}$$

where ③ follows from writing $Z = \bigcup_{i=1}^n z_i^G$, a disjoint union of orbits, and n is of course just the number of orbits on $Z = X \times Y$. $\square$

Remarks. The **disjoint union** or, if you are sophisticated, the **coproduct**, of a family of sets X_i indexed by $i \in I$ is defined as $\coprod_{i \in I} X_i = \{(x, i) \mid x \in X_i\}$. This is to be distinguished from the usual meaning of 'disjoint union' of subsets $A_i \subseteq X$, $i \in I$, where $\bigcup_{i \in I} A_i$ is a disjoint union means that $A_i \cap A_j = \varnothing$ for all $i \neq j$. For instance $\{1, 2, 3\} \amalg \{3, 4\} = \{(1, 0), (2, 0), (3, 0), (3, 1), (4, 1)\}$ (index the first set by 0 and second by 1) whereas $\{1, 2, 3\} \cup \{3, 4\} = \{1, 2, 3, 4\}$. So when $A_i \subseteq X$, $i \in I$, are non-disjoint subsets of X, forming coproduct forces them to be disjoint.

10. (a) Show that the one-dimensional characters of G form a group. Denote this $\widehat{G}$. Show that if $g \in G$, the map $\chi \mapsto \chi(g)$ from $\widehat{G}$ to $\mathbb{C}$ is a character of $\widehat{G}$, hence an element of $\widehat{\widehat{G}}$. What is the kernel of the map $G \to \widehat{\widehat{G}}$.

(b) Show that if G is abelian, the map $G \to \widehat{\widehat{G}}$ is an isomorphism. Conclude from the structure theorem for abelian groups that the groups G and $\widehat{G}$ are isomorphic as abstract groups.
Remark: it is *not* a good idea to identify G and $\widehat{G}$. One reason why is:

(c) Let $\varphi : G \to H$ be a homomorphism of abelian groups. Define a map $\varphi^* : \widehat{H} \to \widehat{G}$. Show φ^* is surjective iff φ is injective.

SOLUTION.

(a) The group operation on $\widehat{G}$ is clearly pointwise multiplication of functions, ie. for $\chi_1, \chi_2 \in \widehat{G}$, the product $\chi_1\chi_2$ is defined by $\chi_1\chi_2(x) := \chi_1(x)\chi_2(x)$ for all $x \in G$. The RHS is just multiplication of complex numbers and so is commutative and associative. $\chi(x) \in \mathbb{C}^\times$ and so the inverse of χ is the map $x \mapsto 1/\chi(x) = \chi(x^{-1})$ which is clearly in $\widehat{G}$ since $\chi^{-1}(xy) = \chi(y^{-1}x^{-1}) = \chi(y^{-1})\chi(x^{-1}) = \chi^{-1}(x)\chi^{-1}(y)$. The trivial character $\mathbb{1}$ is the identity element. So $\widehat{G}$ is an abelian group.

For each $x \in G$, define $\theta_x : \widehat{G} \to \mathbb{C}^\times$, $\chi \to \chi(x)$. $\theta_x \in \widehat{\widehat{G}}$ since $\theta_x(\chi_1\chi_2) = \chi_1(x)\chi_2(x) = \theta_x(\chi_1)\theta_x(\chi_2)$ and $\theta_x(\mathbb{1}) = \mathbb{1}(x) = 1$.

Let $\Theta : G \to \widehat{\widehat{G}}$, $x \mapsto \theta_x$. Then $\ker(\Theta) = \{x \in G \mid \chi(x) = 1 \text{ for all } \chi \in \widehat{G}\} = \bigcap_{\chi \in \widehat{G}} \ker(\chi) = G'$, the commutator subgroup of G, where the last equality follows from (7.1).

(b) If G is abelian, then $\ker(\Theta) = G' = \{1\}$ and so Θ is injective. Since G is finite, Θ is bijective (pigeon hole principle). $\Theta(\chi_1\chi_2) = \chi_1(x)\chi_2(x) = \Theta(\chi_1)\Theta(\chi_2)$. Hence Θ is an isomorphism.

Recall from Example Sheet **1**, Problem **7** that for $k = \mathbb{C}$, every one-dimensional character (there irreducible representation) of a cyclic group $\mathbb{Z}/N$ is of the form $\chi_\xi : \mathbb{Z}/N \to \mathbb{C}^\times$, $n \mapsto \xi^n$ for some Nth root of unity. If we pick $\xi \in \mathbb{C}^\times$ to be any primitive Nth root of unity, say $\xi = e^{2\pi i/N}$, then it's clear that $\chi_\xi^N = \mathbb{1}$ but $\chi_\xi^d \neq \mathbb{1}$ for $d = 1, \dots, N$. Hence $\widehat{\mathbb{Z}/N} = \langle \chi_\xi \rangle \cong \mathbb{Z}/N$. Observe that $\widehat{G_1 \times G_2} = \mathrm{Hom}(G_1 \times G_2, \mathbb{C}^\times) \cong \mathrm{Hom}(G_1, \mathbb{C}^\times) \times \mathrm{Hom}(G_2, \mathbb{C}^\times) = \widehat{G}_1 \times \widehat{G}_2$. Apply this repeatedly to a finite abelian group $G \cong \mathbb{Z}/N_1 \times \cdots \times \mathbb{Z}/N_m$ (structure theorem), we get that $\widehat{G} \cong \widehat{\mathbb{Z}/N_1} \times \dots \widehat{\mathbb{Z}/N_m} \cong \mathbb{Z}/N_1 \times \cdots \times \mathbb{Z}/N_m \cong G$.

(c) Let $\varphi \in \mathrm{Hom}(G, H)$. Define $\varphi^* : \widehat{H} \to \widehat{G}$, $\chi \mapsto \chi \circ \varphi$, the latter is a composition of two homomorphisms with image in $\mathbb{C}^\times$ and so is indeed in $\widehat{G}$. $[\varphi^*(\chi_1\chi_2)](x) = \chi_1\chi_2(\varphi(x)) = \chi_1(\varphi(x))\chi_2(\varphi(x)) = [\varphi^*(\chi_1)\varphi^*(\chi_2)](x)$ and so $\varphi^* \in \mathrm{Hom}(\widehat{H}, \widehat{G})$. If $\varphi : G \to H$ is injective, then we have the exact sequence

$$0 \to G \xrightarrow{\varphi} H.$$

Applying the contravariant functor $\mathrm{Hom}(-, \mathbb{C}^\times)$ gives exact sequence

$$\mathrm{Hom}(H, \mathbb{C}^\times) \xrightarrow{\varphi^*} \mathrm{Hom}(G, \mathbb{C}^\times) \longrightarrow 0$$

and so $\varphi^* : \widehat{H} \to \widehat{G}$ is surjective. Conversely, assuming the previous statement, we have the exact sequence

$$\widehat{H} \xrightarrow{\varphi^*} \widehat{G} \longrightarrow 0.$$

Applying $\mathrm{Hom}(-, \mathbb{C}^\times)$ gives exact sequence

$$0 \longrightarrow \mathrm{Hom}(\widehat{G}, \mathbb{C}^\times) \xrightarrow{(\varphi^*)^*} \mathrm{Hom}(\widehat{H}, \mathbb{C}^\times)$$

and since $\mathrm{Hom}(\widehat{G}, \mathbb{C}^\times) = \widehat{\widehat{G}} \cong G$ by Part (b), $(\varphi^*)^* \simeq \varphi$ is injective. □

11. Compute the character tables of $\mathbf{S}_3, \mathbf{S}_4, \mathbf{S}_5$. Compute the character the character tables of $\mathbf{A}_3, \mathbf{A}_4, \mathbf{A}_5$ ($\mathbf{A}_n \subseteq \mathbf{S}_n$). The groups $\mathbf{S}_n$ act by conjugation on $\mathbf{A}_n$. This induces an action on the set of irreducible representations of $\mathbf{A}_n$. Describe it for $n = 3, 4, 5$.

SOLUTION. See: *Some notes on* $\mathbf{D}_n$, $\mathbf{A}_n$ *and* $\mathbf{S}_n$. □

12. The group $\mathrm{SL}_2(\mathbb{F}_q)$ acts on $\mathbb{P}^1(\mathbb{F}_q) = \mathbb{F}_q \cup \{\infty\}$ by Möbius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}.$$

Show that $\mathrm{SL}_2(\mathbb{F}_q)$ has an irreducible representation of dimension q.

SOLUTION. (By Murray Rogers, Queen's College) If we set $G = \mathrm{SL}_2(\mathbb{F}_q)$ and $X = \mathbb{P}^1(\mathbb{F}_q)$ then we know that for the permutation representation $\mathbb{C}[X]$ the character χ has

$$\langle \chi, \chi \rangle = \#G\text{-orbits on } X \times X.$$

If we define $\Delta = \{(x, x) \in X \times X \mid x \in X\}$, then G acts transitively on Δ, since $g = \left(\begin{smallmatrix} x & -1 \\ 1 & 0 \end{smallmatrix}\right) \in G$ sends (∞, ∞) to (x, x).

Also, G acts transitively on $(X \times X) \backslash \Delta$, since $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ sends $(\infty, 0)$ to (x, y), where

$$a = (x - y)^{-1}x, \qquad b = y, \qquad c = (x - y)^{-1}, \qquad d = 1.$$

Hence G has two distinct orbits on $X \times X$, as these orbits clearly cannot overlap, so $\langle \chi, \chi \rangle = 2$. Thus $\rho_{\mathrm{perm}} = \mathbb{1} \oplus \rho'$, where $\dim(\rho') = |X| - 1 = q$, irreducible. □