Solutions to Example Sheet 1

1. The symmetric group $\mathbf{S}_3$ acts on $\mathbb{R}^2$, by permuting the vertices of an equilateral triangle centered at 0. *Choose* a basis of $\mathbb{R}^2$, and for each $g \in \mathbf{S}_3$, write the matrix of g in this basis.

Solution. Let e_1, e_2, e_3 be the three vertices. Choose any of the two, say, e_1, e_2 to be our basis vectors. Since $e_1 + e_2 + e_3 = 0$ and $\mathbf{S}_3$ acts by permuting the indices, we have:

$$\begin{aligned} (12)e_1 &= 0e_1 + 1e_2 & (13)e_1 &= -1e_1 - 1e_2 & (23)e_1 &= 1e_1 + 0e_2 \\ (12)e_2 &= 1e_1 + 0e_2 & (13)e_2 &= 0e_1 + 1e_2 & (23)e_2 &= -1e_1 - 1e_2 \end{aligned}$$

$$\begin{aligned} (123)e_1 &= 0e_1 + 1e_2 & (132)e_1 &= -1e_1 - 1e_2 \\ (123)e_2 &= -1e_1 - 1e_2 & (132)e_2 &= 1e_1 + 0e_2 \end{aligned}$$

Let ρ denote the permutation representation associated with this action. Then with respect to this basis,

$$\rho(\mathrm{id}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \rho((12)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \rho((13)) = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$\rho((23)) = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \qquad \rho((123)) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \qquad \rho((132)) = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

□

2. Let ρ be a representation of G. Show $\det \rho$ is a one dimensional representation.

Solution. $\det \rho$ is a composition of two homomorphisms and has image in $\mathbb{C}$. □

3. Let $\rho : G \to \mathrm{Aut}(V)$ be a representation of G, and $\chi : G \to k^*$ be a *one* dimensional representation of G. Show

(i) $\widetilde{\rho} : G \to \mathrm{Aut}(V)$, $g \mapsto \chi(g)\rho(g)$ is a representation of G,

(ii) $\widetilde{\rho}$ is irreducible iff ρ is irreducible.

Solution. $\chi(x)$ is a scalar and so commutes with $\rho(y)$ for all $x, y \in G$.

(i) $\widetilde{\rho}(xy) = \chi(xy)\rho(xy) = (\chi(x)\chi(y))(\rho(x)\rho(y)) = (\chi(x)\rho(x))(\chi(y)\rho(y)) = \widetilde{\rho}(x)\widetilde{\rho}(y)$ and $\widetilde{\rho}(\mathrm{id}) = \chi(\mathrm{id})\rho(\mathrm{id}) = I$.

(ii) Let $U \leq V$. $\widetilde{\rho}(x)U \subseteq U$ iff $\chi(x)\rho(x)U \subseteq U$ iff $\rho(x)U \subseteq U$. □

4. Let N be a normal subgroup of G. Given a representation of G/N, define a representation of G and describe which representations of G arise in this way.

Solution. Let the representation be $\rho : G/N \to \mathrm{Aut}(V)$. Let $\pi : G \to G/N$, $x \mapsto xN$ be the quotient map. Then $\rho \circ \pi : G \to \mathrm{Aut}(V)$ is a composition of two homomorphisms, thus a homomorphism and thus a representation of G. [left diagram]

$$\begin{tikzcd} G \arrow[d, "\pi"'] \arrow[dr, "\rho \circ \pi"] & \\ G/N \arrow[r, "\rho"] & \mathrm{Aut}(V) \end{tikzcd} \qquad\qquad \begin{tikzcd} G \arrow[r, "\widetilde{\rho}"] \arrow[d, "\pi"'] & \mathrm{Aut}(V) \\ G/\ker(\widetilde{\rho}) \arrow[ur, "\rho"'] & \end{tikzcd}$$

Let $\widetilde{\rho} : G \to \mathrm{Aut}(V)$ be a representation. Then the map $\rho : G/\ker(\widetilde{\rho}) \to \mathrm{Aut}(V)$, $g\ker(\widetilde{\rho}) \mapsto \widetilde{\rho}(g)$ is a well-defined homomorphism and $\widetilde{\rho} = \rho \circ \pi$. $N = \ker(\widetilde{\rho}) \trianglelefteq G$ is non-trivial precisely when ρ is not faithful. [right diagram] □

5. Let G act on a complex vector space V, and let $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ be a skew-symmetric form, ie. $\langle x, y \rangle = -\langle y, x \rangle$ for all $x, y \in V$. Show

$$(x, y) = \frac{1}{|G|} \sum_{g \in G} \langle gx, gy \rangle$$

is a G-invariant skew-symmetric form. Does this imply every finite subgroup of $\mathrm{GL}_{2n}(\mathbb{C})$ is conjugate to a subgroup of $\mathrm{Sp}_{2n}(\mathbb{C})$?

SOLUTION. Let $h \in G$. Then

$$\begin{aligned}(hx, hy) &= \frac{1}{|G|}\sum_{g\in G}\langle ghx, ghy\rangle = \frac{1}{|G|}\sum_{g'h^{-1}\in G}\langle g'x, g'y\rangle \qquad (\text{write } g = g'h^{-1}) \\ &= \frac{1}{|G|}\sum_{g'\in G}\langle g'x, g'y\rangle = (x, y).\end{aligned}$$

So $(\cdot,\cdot)$ is G-invariant. It is straightforward to check that $(\cdot,\cdot)$ is skew-symmetric and bilinear. It is not true that every finite subgroup of $\mathrm{GL}_{2n}(\mathbb{C})$ is conjugate to a subgroup of $\mathrm{Sp}_{2n}(\mathbb{C}) = \{A \in \mathrm{GL}_{2n}(\mathbb{C}) \mid A^T JA = J\}$ where $J = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0\end{smallmatrix}\right)$. When $n = 1$,

$$\begin{pmatrix} a & c \\ b & d\end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} \iff \begin{pmatrix} ad-bc & 0 \\ 0 & bc-ad\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} \iff ad - bc = 1,$$

so $\mathrm{Sp}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C}) = \{A \in \mathrm{GL}_2(\mathbb{C}) \mid \det A = 1\}$. Since $\det A$ is unchanged under conjugation, we just need to pick an $A \in \mathrm{GL}_2(\mathbb{C})$ that has finite order and $\det A \neq 1$. For instance, $A = \left(\begin{smallmatrix} i & 0 \\ 0 & i\end{smallmatrix}\right)$ generates a subgroup G of order 4 and $\det A = -1$, so $xGx^{-1} \not\subseteq \mathrm{GL}_2(\mathbb{C})$ for any $x \in \mathrm{GL}_2(\mathbb{C})$. $\square$

6. Let G act on a complex vector space V, and let $v \in V$.

(i) Show averaging over the orbit of G on V defines a vector $\overline{v}$ fixed by G.

(ii) If V is irreducible, what can you say about $\overline{v}$?

SOLUTION. Let $v^G = \mathrm{Orb}_G(v) = \{gv \mid g \in G\}$ and $G_v = \mathrm{Stab}_G(v) = \{g \in G \mid gv = v\}$.

(i) Let $n = |v^G| = [G : G_v]$, so $G = \bigcup_{i=1}^n g_iG_v$ (partition into cosets) and $v^G = \{g_1v, \ldots, g_nv\}$. For any $g \in G$, $\bigcup_{i=1}^n gg_iG_v = gG = G = \bigcup_{i=1}^n g_iG_v$. So for each $i = 1, \ldots, n$, there is a unique j such that $gg_iG_v = g_jG_v$. Hence $gv^G = \{gg_1v, \ldots, gg_nv\} = \{g_1v, \ldots, g_nv\} = v^G$. If we define

$$\overline{v} = \frac{1}{|v^G|}\sum_{u\in v^G} gu = \frac{1}{n}\sum_{i=1}^n g_iu,$$

then it follows that

$$g\overline{v} = \frac{1}{n}\sum_{i=1}^n gg_iu = \frac{1}{n}\sum_{i=1}^n g_iu = \overline{v}$$

for all $g \in G$.

(ii) Let $U = \{\lambda\overline{v} \mid \lambda \in \mathbb{C}\} \leq V$. Then $gU = U$ for every $g \in G$ by (i). Since V is irreducible, we must have $U = \{0\}$ or V. Hence either $\overline{v} = 0$ or $V = U$ is one dimensional. $\square$

7. Let $J_{\lambda,n}$ be the $n \times n$ Jordan block with eigenvalue $\lambda \in k$ (k is a field):

$$J_{\lambda,n} = \begin{pmatrix} \lambda & 1 & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \ldots\ldots & & 0 & \lambda \end{pmatrix}_{n\times n},$$

(i) compute $J^r_{\lambda,n}$, for each $r \geq 0$,

(ii) let $G = \mathbb{Z}/N$, and let k be an algebraically closed field of characteristic $p \geq 0$. Determine *all* the representations of G on vector spaces over k, up to equivalence. Which are irreducible? Which do *not* split as a direct sum $W \oplus W'$, with $W \neq 0$ and $W' \neq 0$ (such representations are called "indecomposable").

SOLUTION. We may write $J_{\lambda,n} = \lambda I_n + A_n$ where A_n is the $n \times n$ nilpotent matrix with 1 on the first superdiagonal and 0 everywhere else. Note that for $i \leq n-1$, A_n^i is the matrix with 1 on the ith superdiagonal and 0 everywhere else. $A_n^i = 0$ iff $i \geq n$.

(i) Since λI_n commutes with A_n, we may apply binomial expansion to get

$$J_{\lambda,n}^r = (\lambda I_n + A_n)^r = \lambda^r I_n + \sum_{i=1}^{\min(r,n-1)} \binom{r}{i} \lambda^{r-i} A_n^i.$$

Note that by our observation above, the ith term in the binomial expansion falls on the ith superdiagonal. So if $r \leq n-1$, we have

$$J_{\lambda,n}^r = \begin{pmatrix} \lambda^r & r\lambda^{r-1} & \binom{r}{2}\lambda^{r-2} & \dots & \lambda & 1 & 0 & \dots\dots & 0 \\ 0 & \lambda^r & r\lambda^{r-1} & \dots & r\lambda^2 & \lambda & 1 & 0 \quad \dots & 0 \\ \vdots & \ddots & \ddots & & & & & & \vdots \\ \vdots & & & & & & \ddots & \ddots & \\ 0 & \dots\dots & & & & & 0 & \lambda^r & r\lambda^{r-1} \\ 0 & \dots\dots & & & & & & 0 & \lambda^r \end{pmatrix},$$

and if $r \geq n$, we have

$$J_{\lambda,n}^r = \begin{pmatrix} \lambda^r & r\lambda^{r-1} & \binom{r}{2}\lambda^{r-2} & \dots & \binom{r}{n-1}\lambda^{r-n+1} \\ 0 & \lambda^r & r\lambda^{r-1} & \dots & \binom{r}{n-2}\lambda^{r-n+2} \\ \vdots & \ddots & \ddots & & \\ \vdots & & & & \\ 0 & \dots & 0 & \lambda^r & r\lambda^{r-1} \\ 0 & \dots\dots & & 0 & \lambda^r \end{pmatrix}.$$

(ii) Let $\rho : \mathbb{Z}/N \to \mathrm{Aut}_k(V)$ be a representation of dimension d. Since k is algebraically closed, all eigenvalues $\lambda_1, \dots, \lambda_s$ of the linear map $\rho(1)$ lie in k and so we may pick a Jordan basis for $\rho(1)$, ie. a basis of V such that $\rho(1)$ is a Jordan matrix

$$\rho(1) = \begin{pmatrix} J_{\lambda_1,n_1} & 0 & \dots\dots & 0 \\ 0 & J_{\lambda_2,n_2} & 0 \quad \dots & 0 \\ \vdots & & \ddots & \vdots \\ \vdots & & \ddots & \\ 0 & \dots\dots & 0 & J_{\lambda_s,n_s} \end{pmatrix} \tag{7.1}$$

where each $n_j \geq 0$ and $n_1 + \dots + n_s = d$. Raising (7.1) to the exponent N gives:

$$I_d = \rho(0) = \rho(1)^N = \begin{pmatrix} J_{\lambda_1,n_1}^N & 0 & \dots\dots & 0 \\ 0 & J_{\lambda_2,n_2}^N & 0 \quad \dots & 0 \\ \vdots & & \ddots & \vdots \\ \vdots & & \ddots & \\ 0 & \dots\dots & 0 & J_{\lambda_s,n_s}^N \end{pmatrix}. \tag{7.2}$$

So comparing blocks on both sides for each $j = 1, \dots, s$, we get

$$I_{n_j} = J_{\lambda_j,n_j}^N = \lambda^N I_{n_j} + \sum_{i=1}^{\min(N,n_j-1)} \binom{N}{i} \lambda^{N-i} A_{n_j}^i$$

and we must have:

① $\lambda^N = 1$, ie. λ is an Nth root of unity,

② $B_{n_j} := \sum_{i=1}^{\min(N,n_j-1)} \binom{N}{i} \lambda^{N-i} A_{n_j}^i = 0$.

Let $N = p_1^{\alpha_1} \cdots p_t^{\alpha_t}$ be the prime factorization of N. Then $\mathbb{Z}/N \cong \mathbb{Z}/p_1^{\alpha_1} \times \dots \times \mathbb{Z}/p_t^{\alpha_t}$. We will first get the result for $N = p^\alpha$ and then later use Problem **9** to deduce the result for general N. We need to consider two separate cases:

Case 1. $\mathrm{char}(k) \neq p$

In this case $B_{n_j} \neq 0$: this is clear when $\mathrm{char}(k) = 0$; if $\mathrm{char}(k) = q$ prime $\neq p$, then $\binom{p^\alpha}{1} = p^\alpha \neq 0 \pmod q$ and so the first superdiagonal is non-zero. Hence (7.2) could only hold if $n_1 = \cdots = n_s = 1$ (ie. the B_{n_j} terms cease to exist altogether). So (7.1) becomes

$$\rho(1) = \begin{pmatrix} \lambda_1 & 0 & \ldots\ldots & & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \\ 0 & \ldots\ldots & & 0 & \lambda_s \end{pmatrix}. \tag{7.3}$$

In this case the representations are irreducible iff they are indecomposable and clearly this occurs when they are one dimensional, ie. when $\rho(1) = \lambda_i$ where $\lambda_i^{p^\alpha} = 1$.

Case 2. $\mathrm{char}(k) = p$

Since $p \mid \binom{p^\alpha}{l}$ for $l = 1, \ldots, p^\alpha - 1$, $B_{n_j} = 0$ iff $n_j \leq p^\alpha$. So here the matrix $\rho(1)$ in (7.1) can be non-diagonal. These representations are indecomposable precisely when the matrix in (7.1) is a Jordan block, ie. $\rho(1) = J_{\lambda_i, n_i}$ where $\lambda_i^{p^\alpha} = 1$ and $n_i \in \{1, \ldots, p^\alpha\}$; the irreducible representations among these are the ones with $n_i = 1$, ie. $\rho(1) = \lambda_i$.

Now back to the general $N = p_1^{\alpha_1} \cdots p_t^{\alpha_t}$. Note that Case 2 occurs only when $\mathrm{char}(k) = p_i$ for some $i = 1, \ldots, t$. By Problem **9**, the representations of $\mathbb{Z}/p_1^{\alpha_1} \times \cdots \times \mathbb{Z}/p_t^{\alpha_t}$ are the tensor product representations $\rho_1 \otimes \cdots \otimes \rho_t : \mathbb{Z}/p_1^{\alpha_1} \times \cdots \times \mathbb{Z}/p_t^{\alpha_t} \to \mathrm{GL}_{d_1 \cdots d_t}(\mathbb{C})$, $1 \mapsto \rho_1(1) \otimes \cdots \otimes \rho_t(1)$ where $\rho_i : \mathbb{Z}/p_i^{\alpha_i} \to \mathrm{GL}_{d_i}(\mathbb{C})$ are representations of $\mathbb{Z}/p_i^{\alpha_i}$ of dimension d_i, $i = 1, \ldots, t$. $\rho_1 \otimes \cdots \otimes \rho_t$ is irreducible when each ρ_i is and every irreducible representation of $\mathbb{Z}/p_1^{\alpha_1} \times \cdots \times \mathbb{Z}/p_t^{\alpha_t}$ is of this form.

When $\mathrm{char}(k) \nmid N$, then the irreducible/indecomposable representations of $\mathbb{Z}/N$ are of the form $\mathbb{Z}/N \to \mathbb{C}^\times$, $1 \mapsto \lambda_1 \cdots \lambda_t$ where $\lambda_i^{p_i} = 1$ for $i = 1, \ldots, t$. When $\mathrm{char}(k) \mid N$, may assume WLOG that $\mathrm{char}(k) = p_t$. Then the irreducible representations are of the same form as in the case $\mathrm{char}(k) \nmid N$ while the indecomposable representations are of the form $\mathbb{Z}/N \to \mathrm{GL}_{d_t}(\mathbb{C})$, $1 \mapsto \lambda_1 \cdots \lambda_{t-1} J_{\lambda_t, d_t}$, where $\lambda_i^{p_i} = 1$ for $i = 1, \ldots, t$ and J_{λ_t, d_t} is a Jordan block of dimension $d_t \in \{1, \ldots, p_t^{\alpha_t}\}$. □

Remark. Recall that the tensor product of two matrices $A = (a_{ij}) \in \mathrm{M}(n, \mathbb{F})$ and $B \in \mathrm{M}(m, \mathbb{F})$ is the matrix $A \otimes B \in \mathrm{M}(nm, \mathbb{F})$ defined by

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \ldots & a_{1n}B \\ a_{12}B & a_{22}B & \ldots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \ldots & a_{nn}B \end{pmatrix}$$

where $a_{ij}B$ denotes scalar multiplication, ie. multiplying each entry in B by $a_{ij} \in \mathbb{F}$. For tensor product of $t > 2$ matrices, just apply this $t - 1$ times.

8. (i) Show that if $\rho : G \to \mathrm{GL}_n(\mathbb{R})$ is a homomorphism, then there exists a matrix $P \in \mathrm{GL}_n(\mathbb{R})$ such that $P\rho(g)P^{-1}$ is an orthogonal matrix for each $g \in G$ (recall: A is orthogonal if $AA^T = I$).

(ii) Determine all finite groups which have a faithful representation on a two dimensional *real* vector space.

SOLUTION.

(i) Let $\langle \cdot, \cdot \rangle$ denote the usual inner product on $\mathbb{R}^n$, ie. $\langle u, v \rangle = u^T v$. As usual, we define a G-invariant inner product $(\cdot, \cdot)$ on $\mathbb{R}^n$ by

$$(u, v) := \frac{1}{|G|} \sum_{g \in G} \langle \rho(g)u, \rho(g)v \rangle$$

for all $u, v \in \mathbb{R}^n$. As in Problem **5**, we have $(\rho(g)u, \rho(g)v) = (u, v)$ for all $g \in G$ and it is again straightforward to check that $(\cdot, \cdot)$ is an inner product, ie. a symmetric positive definite bilinear form. Hence there exists a symmetric positive definite matrix $A \in \mathrm{GL}_n(\mathbb{R})$ such that

$$(u, v) = u^T A v$$

for all $u, v \in \mathbb{R}^n$. By the spectral theory for such operators, there exists $Q \in \mathrm{O}(n)$ such that $QAQ^T = D = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ (which are the eigenvalues of A). Let $S = \mathrm{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})$, so $S^2 = D$ and we have $A = Q^T S^2 Q$. Let $g \in G$, so $(\rho(g)u, \rho(g)v) = (u, v)$ gives

$$u^T \rho(g)^T A \rho(g) v = u^T A v$$

for all $u, v \in \mathbb{R}^n$ and hence

$$\rho(g)^T Q^T S^2 Q \rho(g) = Q^T S^2 Q$$

which gives

$$(S^{-1} Q \rho(g)^T Q^T S)(S Q \rho(g) Q^T S^{-1}) = I.$$

Note that S and S^{-1} are both diagonal and thus symmetric. Hence if we let $P = SQ$, then the above equation becomes

$$(P \rho(g) P^{-1})^T (P \rho(g) P^{-1}) = I$$

so $P\rho(g)P^{-1} \in \mathrm{O}(n)$.

(ii) Let G be a finite group and $\rho : G \to \mathrm{GL}_n(\mathbb{R})$ be a faithful representation. By the earlier part, there exists an equivalent representation (which we will denote by the same letter) such that $\rho(G) \leq \mathrm{O}(n)$. Since ρ is faithful, we have $G \cong \rho(G) \leq \mathrm{O}(n)$, ie. G is isomorphic to a (finite) subgroup of $\mathrm{O}(n)$. Hence G is either dihedral or cyclic. □

Remark. One of you pointed out that since every $A \in \mathrm{O}(n)$ has $\det A = \pm 1$ and $\det P\rho(g)P^T = \det \rho(g)$, the result in (i) implies that *every* real representation of a finite group has determinant ± 1. Note that this however does not imply that the original $\rho(g)$ is orthogonal since there are lots of non-orthogonal matrices in $\mathrm{GL}_n(\mathbb{R})$ that have determinant ± 1. That $\det \rho(g) = \pm 1$ follows also from the fact that a linear character must take elements in G to roots of unity, and if ρ is real, then these can only be ± 1.

9. Let $G = \mathbb{Z}/N \times \mathbb{Z}/M$. Determine all the irreducible *complex* representations of G.

SOLUTION. Since every complex irreducible representation of an abelian group is one dimensional (see Remark 2 below for a proof), we have $\rho : \mathbb{Z}/N \times \mathbb{Z}/M \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$. We just need to see where the generators $(1,0), (0,1) \in \mathbb{Z}/N \times \mathbb{Z}/M$ maps to. So suppose $\rho((1,0)) = \lambda \in \mathbb{C}^\times$ and $\rho((0,1)) = \mu \in \mathbb{C}^\times$. Now

$$1 = \rho((0,0)) = \rho((N,0)) = \rho((1,0))^N = \lambda^N$$

so $\lambda \in \{e^{2k\pi i/N} \mid k = 0, \dots, N-1\}$. Likewise, $\mu \in \{e^{2l\pi i/M} \mid l = 0, \dots, M-1\}$. So for $(m,n) \in \mathbb{Z}/N \times \mathbb{Z}/M$,

$$\rho((m,n)) = \rho(m(1,0) + n(0,1)) = \rho((1,0))^m \rho((0,1))^n = \lambda^m \mu^n.$$

Since λ can be any Nth root of unity and μ can be any Mth root of unity, there are altogether $NM = |\mathbb{Z}/N \times \mathbb{Z}/M|$ such irreducible representation. □

Remark 1. This extends easily to $\mathbb{Z}/N_1 \times \cdots \times \mathbb{Z}/N_t$ and since every finite abelian group has a structure of this form, we may construct all irreducible representations of such groups. How about infinite abelian groups? The finitely generated ones would have structure $\mathbb{Z}^r \times T$ where the T is a finite abelian group. Since we know that an irreducible representation of $\mathbb{Z}$ has the form $\mathbb{Z} \to \mathbb{C}^\times$, $n \mapsto \alpha^n$ where $\alpha \in \mathbb{C}^\times$, the same method allows us to construct all the irreducible representations. These are again one dimensional but the number of such representations is now uncountably infinite if $r > 0$ (since we have uncountably infinite choice for $\alpha \in \mathbb{C}$; note that α need not be a root of unity anymore).

Remark 2. Let $\rho : G \to \mathrm{Aut}(V)$ be an irreducible representation of G over a complex vector space V. Let $x \in G$. Then the linear map $\rho(x) : V \to V$, $v \mapsto \rho(x)v$ has eigenvalue $\lambda_x \in \mathbb{C}^\times$. So $V_x := \ker(\rho(x) - \lambda_x \mathbb{1}_V) \neq \{0\}$ since it contains eigenvectors. For $g \in G$ and $v \in V_x$,

$$(\rho(x) - \lambda_x \mathbb{1}_V)\rho(g)v = \rho(xg)v - \rho(g)\lambda_x v = \rho(gx)v - \rho(g)\lambda_x v = \rho(g)(\rho(g) - \lambda_x \mathbb{1}_V)v = 0$$

so $\rho(g)v \in V_x$. Hence V_x is G-invariant and by irreducibility must either be $\{0\}$ (ruled out) or V. It follows that $\rho(x) = \lambda_x \mathbb{1}_V$. Irreducibility again implies that $\rho(x) = \lambda_x$ and V is one dimensional.

10. Let $G = \mathbb{Z}/N$. G acts on $\mathbb{R}^2$ as symmetries of the regular N-gon. Choose a basis of $\mathbb{R}^2$, and write the matrix $\rho(1)$ representing the action of $1 \in \mathbb{Z}/N$ in this basis.

(i) Is this an irreducible representation?

(ii) Regard $\rho(1)$ as a complex matrix, so that we get a representation of $\mathbb{Z}/N$ on $\mathbb{C}^2$. Decompose $\mathbb{C}^2$ into its irreducible summands.

Solution. Let $\rho(1)$ be the anticlockwise rotation through an angle $\frac{2\pi}{N}$. We choose the standard basis $\{(1,0),(0,1)\}$ of $\mathbb{R}^2$ and with respect to this basis,

$$\rho(1) = \begin{pmatrix} \cos\frac{2\pi}{N} & \sin\frac{2\pi}{N} \\ -\sin\frac{2\pi}{N} & \cos\frac{2\pi}{N} \end{pmatrix}.$$

(i) One way to see that ρ is irreducible (over $\mathbb{R}$) is to note that the only proper subspace invariant under the rotation $\rho(1)$ is $\{0\}$. Alternatively, a non-trivial G-invariant subspace of $\mathbb{R}^2$ must be one dimensional and hence we look for $0 \neq v \in \mathbb{R}^2$ such that $\rho(1)v = \lambda v$. The characteristic polynomial is

$$\lambda^2 - 2\cos\tfrac{2\pi}{N}\lambda + 1 = (\lambda - e^{2\pi i/N})(\lambda - e^{-2\pi i/N})$$

and so has no real zeroes (alternatively, $4\cos^2\frac{2\pi}{N} - 4 < 0$). Hence such a v does not exist.

(ii) Over $\mathbb{C}$, the eigenvalues $e^{2\pi i/N}, e^{-2\pi i/N}$ have corresponding eigenvectors $(1,i),(1,-i) \in \mathbb{C}^2$, so the non-trivial G-invariant subspaces are spanned by these and we have

$$\mathbb{C}^2 = \mathbb{C}(1,i) \oplus \mathbb{C}(1,-i).$$

It's a direct sum since $(1,i)$ and $(1,-i)$ are orthogonal with respect to the standard hermitian inner product on $\mathbb{C}^2$. □

11. A hermitian inner product on $\mathbb{C}^2$ is given by a 2×2 matrix X such that $\overline{X}^T = X$; the inner product is $\langle x, y\rangle = x^T X \overline{y}$. *Explicitly* find a hermitian inner product invariant under the group $G \leq \mathrm{GL}_2(\mathbb{C})$ generated by the matrix

$$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$

(Hint: average the standard hermitian inner product).

Solution. It's easy to check that

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \right\}.$$

By the hint, we have

$$\begin{aligned}(u,v) &= \frac{1}{|G|}\sum_{g\in G}\langle gu, gv\rangle \\ &= \frac{1}{3}u^T\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^T\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}^T\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}^T\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}\right]\overline{v} \\ &= u^T\begin{pmatrix} 4/3 & 2/3 \\ 2/3 & 4/3 \end{pmatrix}\overline{v}.\end{aligned}$$

□