A FEW WORDS ON INDUCED REPRESENTATIONS

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Here's another way to look at induced representations. Let $H \leq G$. Then we have a decomposition of G into disjoint union of cosets: $G = g_1H \cup \cdots \cup g_mH$ where $m = [G : H]$. Given $\rho : H \to \mathrm{GL}(n, \mathbb{C})$ a representation of H, extend the definition of ρ to all of G by setting

$$\rho(x) := 0 \quad \text{for every } x \notin H. \tag{1}$$

The **induced representation** $\widetilde{\rho} = \mathrm{Ind}_H^G \rho : G \to \mathrm{GL}(nm, \mathbb{C})$ is defined by

$$\widetilde{\rho}(x) := \begin{pmatrix} \rho(g_1^{-1}xg_1) & \rho(g_1^{-1}xg_2) & \dots & \rho(g_1^{-1}xg_m) \\ \rho(g_2^{-1}xg_1) & \rho(g_2^{-1}xg_2) & \dots & \rho(g_2^{-1}xg_m) \\ \vdots & \vdots & \ddots & \vdots \\ \rho(g_m^{-1}xg_1) & \rho(g_m^{-1}xg_2) & \dots & \rho(g_m^{-1}xg_m) \end{pmatrix}. \tag{2}$$

We can check that (see next page)

$$\widetilde{\rho}(x)\widetilde{\rho}(y) = \widetilde{\rho}(xy) \tag{3}$$

for all $x, y \in G$ and so $\widetilde{\rho}$ is indeed a representation of G of dimension nm.

How does this relate to the characters? Let $\chi : H \to \mathbb{C}$ be the character of ρ, ie. $\chi(x) = \mathrm{tr}\rho(x)$. By (1), we may extend χ to all of G by

$$\chi(x) = \mathrm{tr}\rho(x) = 0 \quad \text{for every } x \notin H.$$

Now the **induced character** $\widetilde{\chi} : G \to \mathbb{C}$ of χ follows from (2):

$$\widetilde{\chi}(x) = \mathrm{tr}\widetilde{\rho}(x) = \sum_{i=1}^{m} \chi(g_i^{-1}xg_i).$$

Given the character table of χ, the more useful formula for computing $\widetilde{\chi}$ would be

$$\boxed{\widetilde{\chi}(x) = \begin{cases} |C_G(x)| \left(\dfrac{\chi(x_1)}{|C_H(x_1)|} + \cdots + \dfrac{\chi(x_r)}{|C_H(x_r)|} \right) & \text{if } x^G \cap H = x_1^H \cup \cdots \cup x_r^H, \\ 0 & \text{if } x^G \cap H = \varnothing. \end{cases}}$$

The first condition simply says that the conjugacy class of x in G when restricted to H splits into a disjoint union of conjugacy classes in H with representatives $x_1, \dots, x_r$.

The proof of (3) on the following page would tell us that the $m \times m$ block matrix $(\rho(g_i^{-1}xg_j))$ in (3) has the following properties:

- in every row i, there is exactly one non-zero block and this lies on the (unique) column j such that $g_i^{-1}xg_j \in H$,
- in every column j, there is exactly one non-zero block and this lies on the (unique) row i such that $g_i^{-1}xg_j \in H$.

This should remind you of the permutation representation. Indeed, G acts on its left coset space $\{g_1H, \ldots, g_mH\}$ by left multiplication and the action simply permutes the cosets. If you find it puzzling that one can get a representation of G out of a representation of a subgroup, which appears to be obtaining more information out of no where, then this should explain it. The induced representation contains information from *two* representations, the original representation of H plus the permutation representation of G on the $\{g_1H, \ldots, g_mH\}$. If for instance, we take $H = \{1\}$, then the induced representation is just the regular representation of G (ie. the permutation representation of G acting on itself by left multiplication).

There are two more things worth noting (both easy to check):

- observe that the definition in (2) involves a choice of coset representatives $g_1, \ldots, g_m$; a different choice would just give an equivalent representation and would thus not change the induced character $\widetilde{\chi}$,
- the block matrix in (2) is invertible, ie. $\widetilde{\rho}$ is indeed a map to $\mathrm{GL}(nm, \mathbb{C})$.

Proof of (3): We need to show that

$$\sum_{k=1}^{m} \rho(g_i^{-1}xg_k)\rho(g_k^{-1}yg_j) = \rho(g_i^{-1}xyg_j)$$

for $i, j = 1, \ldots, m$.

<u>Case 1.</u> $g_i^{-1}xyg_j \notin H$

If so, then $\rho(g_i^{-1}xyg_j) = 0$ by (1). Clearly either $g_i^{-1}xg_k$ or $g_k^{-1}yg_j \notin H$ (otherwise we will get $g_i^{-1}xyg_j = (g_i^{-1}xg_k)(g_k^{-1}yg_j) \in H$). So $\rho(g_i^{-1}xg_k)\rho(g_k^{-1}yg_j) = 0$ for every $k = 1, \ldots, n$. Hence L.H.S. $= 0$.

<u>Case 2.</u> $g_i^{-1}xyg_j \in H$

The element yg_j lies in exactly one coset of the partition $G = g_1H \cup \cdots \cup g_mH$. Let $yg_j \in g_kH$. Then $g_k^{-1}yg_j \in H$ and so $g_i^{-1}xg_k = (g_i^{-1}xyg_j)(g_k^{-1}yg_j)^{-1} \in H$.

For $l \neq k$, since $yg_j \notin g_lH$, we have $g_l^{-1}yg_j \notin H$ and so $\rho(g_l^{-1}yg_j) = 0$. Hence

$$\sum_{l=1}^{m} \rho(g_i^{-1}xg_l)\rho(g_l^{-1}yg_j) = \rho(g_i^{-1}xg_k)\rho(g_k^{-1}yg_j) = \rho(g_i^{-1}xyg_j).$$

$\square$