Solutions to Math 332 Homework 7

5. Since $899 = 29 \times 31$, we have that $28! \equiv -1 \pmod{29}$ and $30! \equiv -1 \pmod{31}$ by Wilson's theorem. The former yields $-1 \times 27! \equiv -1 \pmod{29}$ and so $27! \equiv 1 \pmod{29}$. The latter yields $(-1) \times (-2) \times (-3) \times 27! \equiv -1 \pmod{31}$, so $6 \times 27! \equiv 1 \pmod{31}$ and so $27! \equiv -5 \pmod{31}$. Solve the system

$$\begin{cases} 27! \equiv 1 \pmod{29} \\ 27! \equiv -5 \pmod{31} \end{cases}$$

using Chinese remainder theorem. From

$$31b_1 \equiv 1 \pmod{29}, \qquad 29b_2 \equiv 1 \pmod{31},$$

we get

$$b_1 \equiv 15 \pmod{29}, \qquad b_2 \equiv 15 \pmod{31}.$$

So

$$\begin{aligned} 27! &\equiv 31 \times 15 \times 1 + 29 \times 15 \times (-5) \pmod{29 \times 31} \\ &\equiv 88 \pmod{899}. \end{aligned}$$

14. If $p \mid n$, then clearly $p \mid n^{2p-1} - n$. If $p \nmid n$, then $n^{p-1} \equiv 1 \pmod{p}$ by Fermat's theorem, so $n \times (n^{p-1})^2 \equiv n \times 1^2 \pmod{p}$. Hence $p \mid n^{2p-1} - n$ for every odd prime p. Note that $n^{2p-1} - n$ is always an even number. Hence $2p \mid n^{2p-1} - n$ for every odd prime p.

27. Note that $54 = 2 \times 3^3$. Since $n^{37} - n$ is always an even number, $2 \mid n^{37} - n$ for all n. Now if $(n, 3) = 1$, then as $\phi(3^3) = 18 \mid 36$, Euler's theorem implies that $3^3 \mid n^{36} - 1$ and so $3^3 \mid n^{37} - n$. If $3 \mid n$, then $3 \mid n^{36}$ and so $3 \nmid n^{36} - 1$. Since $n^{37} - n = n(n^{36} - 1)$, we conclude that $p \parallel n^{37} - n$, ie. $3^3 \nmid n^{37} - n$. Hence $54 \mid n^{37} - n$ if and only if $(n, 3) = 1$.

32. The only possible candidate is 1, by 3-16. But 1 clearly doesn't satisfy the condition. So there is no such number.

33. $330 = 2 \times 3 \times 5 \times 11$, so $\phi(330) = 1 \times 2 \times 4 \times 10 = 80$. $857,500 = 2^2 \times 5^4 \times 7^3$, so $\phi(857,500) = (2^2 - 2)(5^4 - 5^3)(7^3 - 7^2) = 294,000$.

34. Factorization yields

$$\begin{aligned} 12! &= (3 \times 4) \times 11 \times (2 \times 5) \times 3^2 \times 2^3 \times 7 \times (2 \times 3) \times 5 \times 2^2 \times 3 \times 2 \\ &= 2^{10} \times 3^5 \times 5^2 \times 7 \times 11, \\ 17! &= 17 \times 2^4 \times (3 \times 5) \times (2 \times 7) \times 13 \times 12! \\ &= 2^{15} \times 3^6 \times 5^3 \times 7^2 \times 11 \times 13 \times 17. \end{aligned}$$

So

$$\begin{aligned} \phi(12!) &= (2^{10} - 2^9)(3^5 - 3^4)(5^2 - 5)(7 - 1)(11 - 1) \\ &= 99,532,800 \\ \phi(17!) &= (2^{15} - 2^{14})(3^6 - 3^5)(5^3 - 5^2)(7^2 - 7)(11 - 1)(13 - 1)(17 - 1) \\ &= 64,210,599,936,000. \end{aligned}$$

39. $\phi(12^k) = 12^{k-1}\phi(12)$ by 3-70.