FINM 331: DATA ANALYSIS FOR FINANCE AND STATISTICS
FALL 2015
PROBLEM SET 1

For this problem set, you should do all problems 'by hand', i.e., without relying on any computer program. You should justify your answers with mathematical arguments — how you get from one step to the next; there will be no credit for just writing down an answer (e.g. if you are asked to find the EVD or SVD of a matrix, just plugging it into MATLAB or Mathematica or R and reproducing the answer on paper earns you precisely zero marks).

1. Verify our claim in the lecture that the condensed SVD of a matrix $A \in \mathbb{R}^{m\times n}$ may be expressed as

$$A = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^\mathsf{T} + \cdots + \sigma_r \mathbf{u}_r \mathbf{v}_r^\mathsf{T}$$

where $r = \operatorname{rank}(A)$, $\mathbf{u}_1, \ldots, \mathbf{u}_r \in \mathbb{R}^m$ and $\mathbf{v}_1, \ldots, \mathbf{v}_r \in \mathbb{R}^n$ are left and right singular vectors of A. Find a similar expression for the EVD of a symmetric matrix $B \in \mathbb{R}^{n\times n}$.

2. Find the singular value decomposition and the Moore–Penrose inverse of the following matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 5 & -1 \\ -1 & 5 \\ 2 & 2 \end{bmatrix}.$$

Verify that $A^\dagger$ indeed satisfies the four defining properties.

3. Let $A, B \in \mathbb{R}^{n\times n}$ and O be the zero matrix. For each of the following statment, either give a proof or a counter example:
(a) If all eigenvalues of A are zero, then $A = O$.
(b) If all singular values of A are zero, then $A = O$.
(c) If $A = A^\mathsf{T}$, then the EVD and SVD of A are identical.
(d) If A and B are similar (i.e., $A = XBX^{-1}$ for some nonsingular X), then A and B have the same eigenvalues.
(e) If A and B have the same eigenvalues, then A and B are similar.
(f) If A and B are similar, then A and B have the same singular values.
(g) The matrices AB and BA always have the same eigenvalues.
(h) The matrices AB and BA always have the same singular values.

4. Let $A \in \mathbb{R}^{n\times n}$. Define

$$\begin{aligned} \rho(A) &= \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}, \\ \omega(A) &= \max\{|\mathbf{x}^\mathsf{T} A \mathbf{x}| : \|\mathbf{x}\|_2 = 1\}, \\ \sigma(A) &= \max\{|\mathbf{x}^\mathsf{T} A^\mathsf{T} A \mathbf{x}|^{1/2} : \|\mathbf{x}\|_2 = 1\}. \end{aligned}$$

(a) Show that

$$\rho(A) \le \omega(A) \le \sigma(A)$$

Can either of the inequalities be strict? Which of these three quantities is always a singular value of A?

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(b) Show that
$$\rho\left(\frac{A+A^\mathsf{T}}{2}\right) \le \sigma(A).$$
Can the inequality be strict?

(c) Show that if $B \in \mathbb{R}^{n\times n}$, then
$$\sigma(AB) \le \sigma(A)\sigma(B),$$
$$\sigma(A+B) \le \sigma(A)+\sigma(B),$$
$$\sigma(A^2-B^2) \le \sigma(A+B)\sigma(A-B).$$

5. Let $A \in \mathbb{R}^{n\times n}$. Consider the matrices
$$A^\mathsf{T}, \quad A^2, \quad \frac{A+A^\mathsf{T}}{2}, \quad A^\mathsf{T}A, \quad AA^\mathsf{T}, \quad \begin{bmatrix} 0 & A \\ A^\mathsf{T} & 0 \end{bmatrix}.$$
Discuss any relations between (i) the eigenvalues, (ii) the singular values, of these matrices and those of A.

6. Let W be a k-dimensional subspace of $\mathbb{R}^n$. Suppose $\mathbf{a}_1, \dots, \mathbf{a}_k \in \mathbb{R}^n$ form a basis for W. Show that the Moore–Penrose inverse of the matrix $A = [\mathbf{a}_1, \dots, \mathbf{a}_k] \in \mathbb{R}^{n\times k}$ is given by
$$A^\dagger = (A^\mathsf{T}A)^{-1}A^\mathsf{T}$$
and deduce a formula for $P_W \in \mathbb{R}^{n\times n}$, the projection onto the subsapce W in terms of A. Is this an orthogonal projection? Find an orthogonal projection matrix for each of the following subspaces:

(a) $W_a = \operatorname{span}\{[1,\dots,1]^\mathsf{T}\}$. Discuss how this is related to the mean
$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i.$$

(b) $W_b = \{[x_1,\dots,x_n]^\mathsf{T} \in \mathbb{R}^n : x_1 + \dots + x_n = 0\}$. Discuss how this is related to the deviations
$$x_i - \overline{x}, \quad i = 1, \dots, n.$$

(c) $W_c = \operatorname{span}\{\mathbf{w}\}$ for a nonzero vector $\mathbf{w} \in \mathbb{R}^n$.

(d) $W_d = W_1 \oplus W_2$ where $W_1 \perp W_2$, i.e., $\mathbf{x}^\mathsf{T}\mathbf{y} = 0$ for all $\mathbf{x} \in W_1$, $\mathbf{y} \in W_2$.

(e) $W_e = E_\lambda = \{\mathbf{x} : B\mathbf{x} = \lambda\mathbf{x}\}$, the λ-eigenspace of a symmetric matrix $B \in \mathbb{R}^{n\times n}$.

(f) $W_f = \operatorname{span}\{[1,1,1,1]^\mathsf{T}, [1,1,0,0]^\mathsf{T}[1,1,1,0]^\mathsf{T}\}$. Here $n = 4$.

7. Let W be a subspace of $\mathbb{R}^n$ and $W^\perp := \{x \in \mathbb{R}^n : \mathbf{x}^\mathsf{T}\mathbf{y} = 0 \text{ for all } \mathbf{y} \in W\}$ be its orthogonal complement. Since $\mathbb{R}^n = W \oplus W^\perp$, we may define $P : \mathbb{R}^n \to \mathbb{R}^n$ by $P\mathbf{v} = \mathbf{w}$ where $\mathbf{v} = \mathbf{w} + \mathbf{w}'$ with $\mathbf{w} \in W$ and $\mathbf{w}' \in W^\perp$. We will see that this gives another way to define projection onto W.

(a) Show that P is an orthogonal projection and $\operatorname{im}(P) = W$.

(b) Show that such a P is uniquely determined by W.

(c) Show that for every $\mathbf{v} \in \mathbb{R}^n$, $\mathbf{v}^\mathsf{T}P\mathbf{v} \ge 0$.

(d) Show that for every $\mathbf{v} \in \mathbb{R}^n$, $\|P\mathbf{v}\|_2 \le \|\mathbf{v}\|_2$.

(e) Show that $I - P$ is the orthogonal projection onto $W^\perp$.

(f) Show that for every $\mathbf{v} \in \mathbb{R}^n$,
$$\|\mathbf{v}\|_2^2 = \|P\mathbf{v}\|_2^2 + \|(I-P)\mathbf{v}\|_2^2.$$

(g) Show that P is similar to a diagonal matrix of the form $\operatorname{diag}(1,\dots,1,0,\dots,0)$ where the number of 1's equals $\dim W$.