For this problem set, you should do all problems ‘by hand’, i.e., without relying on any computer program. You should justify your answers with mathematical arguments — how you get from one step to the next; there will be no credit for just writing down an answer (e.g. if you are asked to find the EVD or SVD of a matrix, just plugging it into MATLAB or Mathematica or R and reproducing the answer on paper earns you precisely zero marks).

1. Verify our claim in the lecture that the condensed SVD of a matrix $A \in \mathbb{R}^{m \times n}$ may be expressed as

$$A = \sigma_1 u_1 v_1^T + \cdots + \sigma_r u_r v_r^T$$

where $r = \text{rank}(A)$, $u_1, \ldots, u_r \in \mathbb{R}^m$ and $v_1, \ldots, v_r \in \mathbb{R}^n$ are left and right singular vectors of $A$. Find a similar expression for the EVD of a symmetric matrix $B \in \mathbb{R}^{n \times n}$.

2. Find the singular value decomposition and the Moore–Penrose inverse of the following matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & -1 \\ -1 & 5 \\ 2 & 2 \end{bmatrix}.$$  

Verify that $A^\dagger$ indeed satisfies the four defining properties.

3. Let $A, B \in \mathbb{R}^{n \times n}$ and $O$ be the zero matrix. For each of the following statement, either give a proof or a counter example:

   (a) If all eigenvalues of $A$ are zero, then $A = O$.

   (b) If all singular values of $A$ are zero, then $A = O$.

   (c) If $A = A^\top$, then the EVD and SVD of $A$ are identical.

   (d) If $A$ and $B$ are similar (i.e., $A = XBX^{-1}$ for some nonsingular $X$), then $A$ and $B$ have the same eigenvalues.

   (e) If $A$ and $B$ are similar, then $A$ and $B$ have the same singular values.

4. Let $W$ be a $k$-dimensional subspace of $\mathbb{R}^n$. Suppose $a_1, \ldots, a_k \in \mathbb{R}^n$ form a basis for $W$. Show that the Moore–Penrose inverse of the matrix $A = [a_1, \ldots, a_k] \in \mathbb{R}^{n \times k}$ is given by

$$A^\dagger = (A^\top A)^{-1} A^\top$$

and deduce a formula for $P_W \in \mathbb{R}^{n \times n}$, the projection onto the subspace $W$ in terms of $A$. Is this an orthogonal projection? Find an orthogonal projection matrix for each of the following subspaces:

   (a) $W_a = \text{span}\{[1, \ldots, 1]^\top\}$. Discuss how this is related to the mean

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i.$$ 

   (b) $W_b = \{[x_1, \ldots, x_n]^\top \in \mathbb{R}^n : x_1 + \cdots + x_n = 0\}$. Discuss how this is related to the deviations

$$x_i - \bar{x}, \quad i = 1, \ldots, n.$$
(c) \( W_c = \text{span}\{w\} \) for a nonzero vector \( w \in \mathbb{R}^n \).

(d) \( W_d = W_1 \oplus W_2 \) where \( W_1 \perp W_2 \), i.e., \( x'y = 0 \) for all \( x \in W_1, y \in W_2 \).

(e) \( W_e = E_{\lambda} = \{x : Bx = \lambda x\} \), the \( \lambda \)-eigenspace of a symmetric matrix \( B \in \mathbb{R}^{n \times n} \).

(f) \( W_f = \text{span}\{[1, 1, 1, 1]^T, [1, 1, 0, 0]^T, [1, 1, 0, 0]^T\} \). Here \( n = 4 \).

5. Let \( W \) be a subspace of \( \mathbb{R}^n \) and \( W^\perp := \{x \in \mathbb{R}^n : x'y = 0 \text{ for all } y \in W\} \) be its orthogonal complement. Since \( \mathbb{R}^n = W \oplus W^\perp \), we may define \( P : \mathbb{R}^n \to \mathbb{R}^n \) by \( Pv = w \) where \( v = w + w' \) with \( w \in W \) and \( w' \in W^\perp \). We will see that this gives another way to define projection onto \( W \).

(a) Show that \( P \) is an orthogonal projection and \( \text{im}(P) = W \).

(b) Show that such a \( P \) is uniquely determined by \( W \).

(c) Show that for every \( v \in \mathbb{R}^n \), \( v^TPv \geq 0 \).

(d) Show that for every \( v \in \mathbb{R}^n \), \( \|Pv\|_2 \leq \|v\|_2 \).

(e) Show that \( I - P \) is the orthogonal projection onto \( W^\perp \).

(f) Show that for every \( v \in \mathbb{R}^n \),
\[
\|v\|_2^2 = \|Pv\|_2^2 + \|(I-P)v\|_2^2.
\]

(g) Show that \( P \) is similar to a diagonal matrix of the form \( \text{diag}(1, \ldots, 1, 0, \ldots, 0) \) where the number of 1’s equals \( \dim W \).

6. Let \( x_1, \ldots, x_n \in \mathbb{R}^p \), \( n \geq p \), and set
\[
X := \begin{bmatrix}
    x_1^T \\
    x_2^T \\
    \vdots \\
    x_n^T
\end{bmatrix} = \begin{bmatrix}
    x_{11} & x_{12} & \cdots & x_{1p} \\
    x_{21} & x_{22} & \cdots & x_{2p} \\
    \vdots & \vdots & \ddots & \vdots \\
    x_{n1} & x_{n2} & \cdots & x_{np}
\end{bmatrix} \in \mathbb{R}^{n \times p}.
\]

Suppose that \( X^T1 = 0 \). Let the eigenvalue decomposition of the matrix \( S := \frac{1}{n}X^TX \in \mathbb{R}^{p \times p} \) be given by
\[
S = V\Lambda V^T, \quad V = [v_1, \ldots, v_p], \quad V^TV = I, \quad \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_p), \quad \lambda_1 \geq \cdots \geq \lambda_p.
\]
We would like to plot the projections of \( x_1, \ldots, x_p \) onto the 2-dimensional subspace \( W := \text{span}\{v_j, v_k\} \subseteq \mathbb{R}^p \), i.e., on a graph whose \( x \)-axis is \( v_j \) and \( y \)-axis is \( v_k \).

(a) Write down an expression for \( P_W \in \mathbb{R}^{p \times p} \), the orthogonal projection onto \( W \). Simplify your expression as much as possible.

(b) Show that the eigenvectors of \( S \) are the right singular vectors of \( X \).

(c) Show that to plot the projections of \( x_1, \ldots, x_n \in \mathbb{R}^p \) onto \( W \cong \mathbb{R}^2 \), we may simply plot the \( n \) points
\[
\{(\sigma_j u_{ij}, \sigma_k u_{ik}) \in \mathbb{R}^2 : i = 1, \ldots, n\}
\]
where \( U = [u_{ij}] \in \mathbb{R}^{n \times n} \) and \( \Sigma = \text{diag}(\sigma_1, \ldots, \sigma_p) \in \mathbb{R}^{n \times p} \) are the matrix of left singular vectors and matrix of singular values respectively.