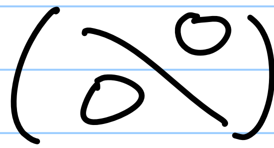
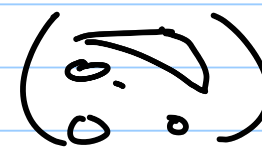
Iterative Methods — April 10, 2006

Note Title 4/10/2006

$$A = D - E - F$$

D: , E: , F: 

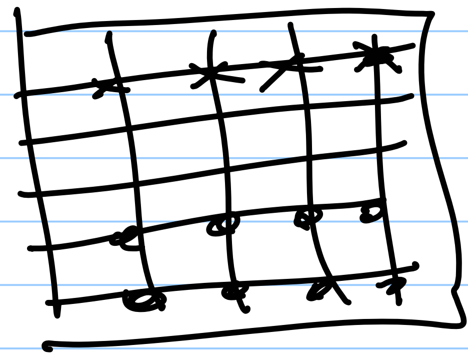
$$D\underset{\sim}{x}^{k+1} = \omega\left(E\underset{\sim}{x}^{k+1} + F\underset{\sim}{x}^{k} + b\right) + (1-\omega)\underset{\sim}{x}^{k}$$

$\omega = ?$

$\omega > 1$ over relaxation

$0 < \omega < 1$ under relaxation

S.S.O.R. (Symmetric S.O.R.)



$$\rightarrow \quad (D - \omega E)\, x^{k+1/2} = (\omega F + (1-\omega) D)\, \underset{\sim}{x}^{k} + \omega \underset{\sim}{b}$$

$$(D - \omega F)\, x^{k+1} = (\omega E + (1-\omega) D)\, x^{k+1/2} + \omega \underset{\sim}{b}$$

$$G_\omega = (D - \omega F)^{-1} (\omega E + (1-\omega) D) (D - \omega E)^{-1} (\omega F + (1-\omega) D)$$

$$\omega = ? \quad , \quad \underset{\sim}{e}^{k+1} = G_\omega \underset{\sim}{e}^{(k)}$$

$$\underset{\sim}{e}^k = G_\omega^k \underset{\sim}{e}^0$$

Convergence ? Implementation ?

Jacobi: $M_J = D^{-1}(E+F)$

$$\underset{\sim}{e}^k = M^k \underset{\sim}{e}^0, \quad \underset{\sim}{e}^{(k)} = M \underset{\sim}{e}^{k-1}$$

$$\|\underset{\sim}{e}^k\| \leq \|M\| \, \|\underset{\sim}{e}^{k-1}\|$$

If $\|M\| < 1$, convergence

$$\|M_J\|_\infty = \max \sum_{j \neq i} \left|\frac{a_{ij}}{a_{ii}}\right| = \rho < 1$$

G.S. $\quad M_{GS} = (D-E)^{-1} F$

$\|M\|_\infty < 1 \Rightarrow$ convergence G.S.

$A = A^T$, A p.d. convergence G.S.

S.O.R. $\quad \underline{A = A^T, \text{ p.d.}}$

$$\mathcal{L}_\omega = (D - \omega E)^{-1} (\omega F + (1-\omega) D)$$

$$A := D^{-1/2} A D^{-1/2} = \begin{pmatrix} I & \xi \\ \xi & \end{pmatrix}, \quad \mathcal{D}(A) = I$$

$$\prod_{i=1}^{n} \lambda_i(\mathcal{L}_\omega) = \det(\mathcal{L}_\omega)$$

$$\mathcal{L}_\omega = (I - \omega E)^{-1}(\omega F + (1-\omega) I)$$

$$\det(\mathcal{L}_\omega) = \det(I - \omega E)^{-1} \cdot \det(\omega F + (1-\omega) I)$$

$$= 1 \cdot (1-\omega)^n \qquad 0 < \omega < 2$$

S.O.R converges when $A = A^T$, p.d

when $0 < \omega < 2$.

David Young.

$$\Pi A \Pi^T = \begin{pmatrix} I & F \\ F^T & I \end{pmatrix}$$

"Property A"

Red / Black

$$\hat{\omega} = \frac{2}{1+\sqrt{1-\|F\|_2^2}} > 1$$

$$\rho(\mathcal{L}_\omega) = \max_i |\lambda_i(\mathcal{L}_\omega)|$$

$\min_\omega \rho(\mathcal{L}_\omega)$: <u>optimal ω</u>

$$\underset{\sim}{e}^k = \mathcal{L}_\omega e^{k-1} = \mathcal{L}_\omega^k \underset{\sim}{e}^0$$

$\|\underset{\sim}{e}^k\| \leq \|\mathcal{L}_\omega^k\| \, \|\underset{\sim}{e}^0\|.$ $\quad \min \|\mathcal{L}_\omega^k\|$

When $\omega = \hat{\omega}$, all eigenvalues of $\mathcal{L}_\omega$ are equal in magnitude

$$\rho(\mathcal{L}_{\hat{\omega}}) = |\hat{\omega} - 1|$$

$$\|\mathcal{L}_\omega^k\| \le \|\mathcal{L}_\omega\|^k \sim [\rho(\mathcal{L}_\omega)]^k \cdot (\quad)$$

$$M \underset{\sim}{x}^{k+1} = N \underset{\sim}{x}^k + \underset{\sim}{b}$$

$$M \underset{\sim}{e}^{k+1} = N \underset{\sim}{e}^k \quad , \quad \underset{\sim}{e}^k = (\underset{\sim}{x} - \underset{\sim}{x}^k)$$

$$\underset{\sim}{e}^{k+1} = M^{-1} N \underset{\sim}{e}^k \equiv B \underset{\sim}{e}^k$$

$$\underset{\sim}{e}^{k} = B^{k} \underset{\sim}{e}^{0}$$

Jordan Canonical Form

$$B = Q J Q^{-1}$$

$$J = \begin{pmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \ddots & \\ 0 & & & J_p \end{pmatrix}, \quad B^k = Q J^k Q^{-1}$$

$$J^k = \begin{pmatrix} J_1^k & & 0 \\ & \ddots & \\ 0 & & J_p^k \end{pmatrix}$$

$$J_\ell^k = \begin{pmatrix} \lambda^k & \binom{k}{1}\lambda^{k-1} & \binom{k}{2}\lambda^{k-2} & \\ & \ddots & \ddots & \ddots \\ & & \ddots & \\ & & & \lambda^k \end{pmatrix}$$

$$J_\ell = \begin{pmatrix} \lambda_\ell & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_\ell \end{pmatrix}$$

$$J_\ell^k = (\lambda_\ell I + K)^k = \sum_{j=0}^{k} \binom{k}{j} \lambda_\ell^j K^{k-j}$$

$$K = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$$

$|\lambda_1| \geq |\lambda_2| \geq \dots |\lambda_\ell|$

Theorem $\|B^k\|_2 \lesssim \|Q\| \cdot \|Q^{-1}\| \cdot \rho^k \binom{k}{n_1}$

$\rho = |\lambda_1|$

$$\begin{pmatrix} \rho & 1 \\ 0 & \rho \end{pmatrix}^k = \begin{pmatrix} \rho^k & k\rho^{k-1} \\ 0 & \rho^k \end{pmatrix}$$

$$E \sim L, \quad F \sim U$$

S.S.O.R.

$$G_\omega = (I + \omega U)^{-1}\left(-\omega L + (1-\omega) I\right)$$

$$\times \, (I + \omega L)^{-1}\left(-\omega U + (1-\omega) I\right)$$

$\lambda(AB) = \lambda(BA)$

$$= (I + \omega U)^{-1} (I + \omega L)^{-1} \left(-\omega L + (1-\omega) I\right)$$

$$\times \left(-\omega U + (1-\omega) I\right)$$

$$G_\omega \underset{\sim}{z} = \lambda \underset{\sim}{z}$$

$$\left(-\omega L + (1-\omega) I\right)\left(-\omega U + (1-\omega) I\right) z = \lambda (I + \omega L)(I + \omega U) z$$

$$\left(B^{-1} A z = \lambda z \equiv A z = \lambda B z \right)$$

$$\lambda = \frac{\underset{\sim}{z}^T(-\omega L + (1-\omega) I)(-\omega U + (1-\omega) I)\underset{\sim}{z}}{\underset{\sim}{z}^T (I+\omega L)(I+\omega U)\underset{\sim}{z}}$$

After some manipulations

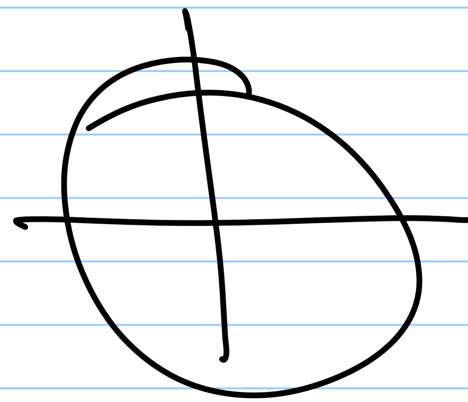
$$= 1 - (2-\omega)\omega \left\{ \frac{\overset{\geq 0 ?}{\underset{\sim}{z}^T A \underset{\sim}{z}}}{\underset{\sim}{z}^T (I+\omega L)(I+\omega U)\underset{\sim}{z}} \right\} \geq 0$$

for all $\boxed{0 < \omega < 2.}$

So G_ω is similar to

$$(I+\omega L)^{-1}\underbrace{((1-\omega)I - \omega L)}_{K^T}\underbrace{((1-\omega)I - \omega U)(I+\omega U)^{-1}}_{K}$$

The eigenvalues of G_ω are real & positive.

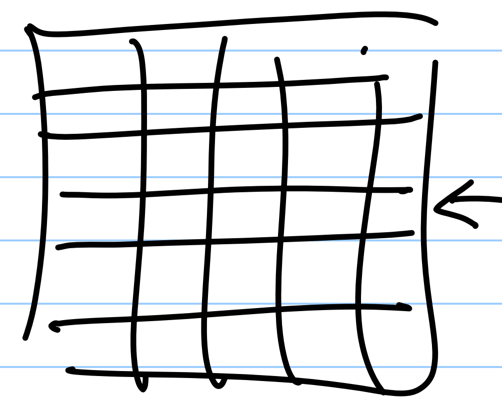


No acceleration of S.O.R.

S.S.O.R. can be accelerated because the eigenvalues of G_ω are real and positive.

BLOCK METHODS

$$A = \begin{pmatrix} A_1 & B_1 & & \\ B_1^T & \ddots & \ddots & \\ & \ddots & \ddots & B_{\ell-1} \\ & & B_{\ell-1}^T & A_\ell \end{pmatrix}$$



Poisson's equation in 2-D.

$$B_{i-1}^T \underset{\sim}{x}_{i-1}^{(k+1)} + A_i \underset{\sim}{x}_i^{k+1} + B_i \underset{\sim}{x}_{i+1}^{k} = \underset{\sim}{f}_i$$

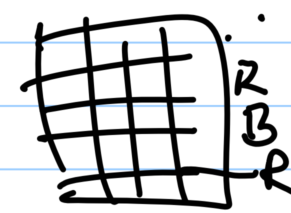
BLOCK JACOBI, BLOCK G-S.

$$M_{BJ} = \begin{pmatrix} A_1^{-1} & & 0 \\ & \ddots & \\ 0 & & A_k^{-1} \end{pmatrix} \begin{pmatrix} 0 & B_1 & & \\ B_1^T & 0 & \ddots & \\ & \ddots & \ddots & B_{k-1} \\ & & B_{k-1}^T & 0 \end{pmatrix}$$

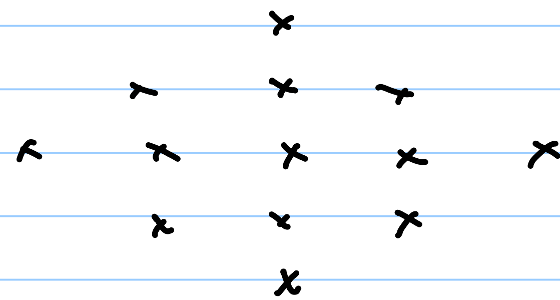
$$\| M_{BJ} \|$$

$$\| M_J \| = \max \sum_{i \neq j} \left| \frac{a_{ij}}{a_{ii}} \right| < 1$$

Re-order equations Red/Black



Biharmonic 4^{th} order.



$$\begin{pmatrix} A_1 & B_1 & C_1 & & \\ B_1^T & A_2 & B_2 & C_2 & \\ C_1^T & B_2^T & A_3 & B_3 & C_3 \\ & & & & \end{pmatrix}$$