In general, a semi-iterative method is one that comprises two steps:

\[ x^{(k+1)} = Mx^{(k)} + b \] (Iteration)

and

\[ y^{(m)} = \sum_{k=0}^{m} \alpha_k^{(m)} x^{(k)}. \] (Extrapolation)

As in the lectures, we will assume that \( M = I - A \) with \( \rho(M) < 1 \) and that we are interested to solve \( Ax = b \) for some nonsingular matrix \( A \in \mathbb{C}^{n \times n} \). Let

\[ e^{(k)} = x^{(k)} - x \quad \text{and} \quad \epsilon^{(m)} = y^{(m)} - x. \]

(a) By considering what happens when \( x^{(0)} = x \), show that it is natural to impose

\[ \sum_{k=0}^{m} \alpha_k^{(m)} = 1 \] (1.1)

for all \( m \in \mathbb{N} \cup \{0\} \). Henceforth, we will assume that (1.1) is satisfied for all problems in this problem set.

(b) Show that for all \( m \in \mathbb{N} \), we may write

\[ \epsilon^{(m)} = P_m(M)e^{(0)} \]

for some \( P_m \in \mathbb{C}[x] \) with \( \deg(P_m) = m \) and \( P_m(1) = 1 \).

(c) Hence deduce that a necessary condition for convergence is that

\[ \lim_{m \to \infty} \|P_m(M)\|_2 < 1 \]

where \( \| \cdot \|_2 \) is the spectral norm. Is this condition also sufficient?

(d) Consider the case when

\[ \alpha_0^{(m)} = \alpha_1^{(m)} = \cdots = \alpha_m^{(m)} = \frac{1}{m + 1} \]

for all \( m \in \mathbb{N} \cup \{0\} \). Show that if

\[ \lim_{k \to \infty} x^{(k)} = x \]

then

\[ \lim_{m \to \infty} y^{(m)} = x. \]

Is the converse also true?

2. It is clear that in any semi-iterative method defined by some \( M \in \mathbb{C}^{n \times n} \) with \( \rho(M) < 1 \), we would like to solve the problem

\[ \min_{P \in \mathbb{C}[x], \deg(P)=m, P(1)=1} \|P(M)\|_2. \] (2.2)

Note that in the lectures, we required the polynomial \( P \) to be monic. Here we use a different condition, \( P(1) = 1 \), motivated by Problem 1(a).

Date: January 21, 2012 (Version 1.3); due: January 26, 2012.
(a) Show that if \( m \geq n \), then a solution to (2.2) is given by
\[
P_m(x) = \frac{x^{m-n} \det(xI - M)}{\det(I - M)}.
\]
How do we know that the denominator is non-zero?

(b) From now on assume that \( M \) is Hermitian with minimum and maximum eigenvalues \( \lambda_{\min}, \lambda_{\max} \in \mathbb{R} \). Define
\[
\|f\|_\infty = \sup_{x \in [\lambda_{\min}, \lambda_{\max}]} |f(x)|.
\]
Emulating our discussions in the lectures, show that for \( m = 0, 1, \ldots, n - 1 \), the solution to the relaxed problem
\[
\min_{P \in \mathbb{C}[x], \deg(P) = m, P(1) = 1} \|P\|_\infty
\]
would yield an upper bound to (2.2).

(c) Again by emulating our discussions in the lectures, show that the solution to (2.3) for \( \lambda_{\min} = -1 \) and \( \lambda_{\max} = +1 \) is given by the Chebyshev polynomials,
\[
C_m(x) = \begin{cases} 
\cos(m \cos^{-1} x) & |x| \leq 1, \\
\cosh(m \cosh^{-1} x) & |x| \geq 1.
\end{cases}
\]

(d) Hence deduce that the solution to (2.3) for \( \lambda_{\min} = a \) and \( \lambda_{\max} = b \) is given
\[
P_m(x) = \frac{C_m\left(\frac{2x - (b + a)}{b - a}\right)}{C_m\left(\frac{2 - (b + a)}{b - a}\right)}.
\]
Note that this solves (2.3) for all \( m \in \mathbb{N} \) and not just \( m \leq n - 1 \).

(e) Show that the solution in (d) is unique.

3. Let \( M \in \mathbb{C}^{n \times n} \) be Hermitian with \( \rho(M) = \rho < 1 \). Moreover, suppose that
\[
\lambda_{\min} = -\rho, \quad \lambda_{\max} = \rho.
\]

(a) Show that the \( P_m \)'s in (2.4) satisfy a three-term recurrence relation
\[
C_{m+1}\left(\frac{1}{\rho}\right) P_{m+1}(x) = \frac{2x}{\rho} C_m\left(\frac{1}{\rho}\right) P_m(x) - C_{m-1}\left(\frac{1}{\rho}\right) P_{m-1}(x)
\]
for all \( m \in \mathbb{N} \).

(b) Show that the semi-iterative method with \( \alpha_k^{(m)} \) given by the coefficient of \( P_m \) in (2.4) may be written as
\[
y^{(m+1)} = \omega_{m+1}(My^{(m)} - y^{(m-1)} + b) + y^{(m-1)}
\]
where \( \omega_1 = 1 \) and
\[
\omega_{m+1} = \frac{2C_m(1/\rho)}{\rho C_{m+1}(1/\rho)}
\]
for \( m = 0, 1, 2, \ldots \). This is a slightly different Chebyshev method where we choose the normalization (1.1) instead of \( \alpha_k^{(m)} = 1 \) in the lecture.

(c) Show that
\[
\|P_m(M)\|_2 = \frac{1}{C_m(1/\rho)} = \frac{1}{\cosh(m\sigma)}
\]
where \( \sigma = \cosh^{-1}(1/\rho) \). Deduce that \( \|P_m(M)\|_2 \) is a strictly decreasing sequence for all \( m = 0, 1, 2, \ldots \).
(d) Show that
\[ e^{-\sigma} = (\omega - 1)^{1/2} \]
where
\[ \omega = \frac{2}{1 + \sqrt{1 - \rho^2}} \]  
(3.5)
and deduce that
\[ \|P_m(M)\|_2 = \frac{2(\omega - 1)^{m/2}}{1 + (\omega - 1)^m}. \]
(e) Hence show that \( (\omega_m)_{m=0}^\infty \) is strictly decreasing for \( m \geq 2 \) and that
\[ \lim_{m \to \infty} \omega_m = \omega. \]

4. Let \( M \in \mathbb{C}^{n \times n} \) be nonsingular with \( \rho(M) < 1 \) and suppose we are interested in solving \( Mx = b \).
(a) Show that SOR applied to the system
\[
\begin{bmatrix}
I & -M \\
-M & I
\end{bmatrix}
\begin{bmatrix}
x \\
z
\end{bmatrix}
= \begin{bmatrix}
b \\
b
\end{bmatrix}
\]
yields the following iterations
\[ x^{(m+1)} = \omega(Mz^{(m)} - x^{(m)} + b) + x^{(m)}, \]
\[ z^{(m+1)} = \omega(Mx^{(m+1)} - z^{(m)} + b) + z^{(m)}, \]
for \( m = 0, 1, 2, \ldots \).
(b) Define the sequence of iterates \( y^{(m)} \) by
\[ y^{(m)} = \begin{cases} 
x^{(k)} & \text{if } m = 2k, \\
z^{(k)} & \text{if } m = 2k + 1. 
\end{cases} \]
Show that the iterations obtained in (a) are exactly the iterations in Problem 3(b). This shows that SOR is equivalent to Chebyshev but with \( \omega_m = \omega \) for all \( m \in \mathbb{N} \). Note that if \( \omega \) is chosen to be the value in (3.5), then this is in fact the optimal SOR parameter (cf. Section 10 in the lecture notes on Stationary Methods).