STAT 309: MATHEMATICAL COMPUTATIONS I
FALL 2021
PROBLEM SET 4

1. Let $A \in \mathbb{R}^{m\times n}$ where $m \geq n$ and $\operatorname{rank}(A) = n$. Suppose GECP is performed on A to get

$$\Pi_1 A \Pi_2 = LU$$

where $L \in \mathbb{R}^{m\times n}$ is unit lower triangular, $U \in \mathbb{R}^{n\times n}$ is upper triangular, and $\Pi_1 \in \mathbb{R}^{m\times m}$, $\Pi_2 \in \mathbb{R}^{n\times n}$ are permutation matrices.

(a) Show that U is nonsingular and that L is of the form

$$L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$$

where $L_1 \in \mathbb{R}^{n\times n}$ is nonsingular.

(b) We will see how the LU factorization may be used to solve the least squares problem

$$\min_{\mathbf{x}\in\mathbb{R}^n} \|A\mathbf{x} - \mathbf{b}\|_2.$$

(i) Show that the problem may be solved via

$$U\widetilde{\mathbf{x}} = \mathbf{y}, \quad L^{\mathsf{T}} L\mathbf{y} = L^{\mathsf{T}}\widetilde{\mathbf{b}},$$

where $\widetilde{\mathbf{b}} = \Pi_1\mathbf{b}$ and $\widetilde{\mathbf{x}} = \Pi_2^{\mathsf{T}}\mathbf{x}$.

(ii) Describe how you would compute the solution $\mathbf{y}$ in

$$L^{\mathsf{T}} L\mathbf{y} = L^{\mathsf{T}}\widetilde{\mathbf{b}}.$$

2. Let $\varepsilon > 0$. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1+\varepsilon \\ 1 & 1-\varepsilon \end{bmatrix}.$$

(a) Why is it a bad idea to solve the normal equation associated with A, i.e.

$$A^{\mathsf{T}} A\mathbf{x} = A^{\mathsf{T}}\mathbf{b}$$

when ε is small?

(b) Show that the condensed LU factorization of A is

$$A = LU = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & \varepsilon \end{bmatrix}.$$

(c) Why is it a much better idea to solve the normal equation associated with L, i.e.

$$L^{\mathsf{T}} L\mathbf{y} = L^{\mathsf{T}}\widetilde{\mathbf{b}}?$$

This shows that the method in Problem **1** is a more stable method than using the normal equation in (a) directly.

(d) Show that the Moore–Penrose pseudoinverse of A is

$$A^{\dagger} = \frac{1}{6}\begin{bmatrix} 2 & 2-3\varepsilon^{-1} & 2+3\varepsilon^{-1} \\ 0 & 3\varepsilon^{-1} & -3\varepsilon^{-1} \end{bmatrix}.$$

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(e) Describe a method to compute $A^\dagger$ given L and U. Verify that your method is correct by checking it against the expression in (d).

3. We will now discuss an alternative method to solve the least squares problem in Problem **1** that is more efficient when $m-n<n$.

(a) Show that the least squares problem in Problem **1** is equivalent to

$$\min_{\mathbf{z}\in\mathbb{R}^n}\left\|\begin{bmatrix} I_n \\ S\end{bmatrix}\mathbf{z}-\widetilde{\mathbf{b}}\right\|_2$$

where $S=L_2L_1^{-1}$ and $L_1\mathbf{y}=\mathbf{z}$. Here and below, I_n denotes the $n\times n$ identity matrix.

(b) Write

$$\widetilde{\mathbf{b}}=\begin{bmatrix}\widetilde{\mathbf{b}}_1\\ \widetilde{\mathbf{b}}_2\end{bmatrix}$$

where $\widetilde{\mathbf{b}}_1\in\mathbb{R}^n$ and $\widetilde{\mathbf{b}}_2\in\mathbb{R}^{m-n}$. Show that the solution $\mathbf{z}$ is given by

$$\mathbf{z}=\widetilde{\mathbf{b}}_1+S^{\mathsf{T}}(I_{m-n}+SS^{\mathsf{T}})^{-1}(\widetilde{\mathbf{b}}_2-S\widetilde{\mathbf{b}}_1).$$

(c) Explain why when $m-n<n$, the method in (a) is much more efficient than the method in Problem **1**. For example, what happens when $m=n+1$?

4. Let $\mathbf{c}\in\mathbb{R}^n$ and consider the linearly constrained least squares problem/minimum norm linear system

$$\begin{aligned} &\text{minimize} && \|\mathbf{w}\|_2\\ &\text{subject to} && A^{\mathsf{T}}\mathbf{w}=\mathbf{c}.\end{aligned}$$

(a) If we write $\widetilde{\mathbf{c}}=\Pi_2^{\mathsf{T}}\mathbf{c}$ and $\widetilde{\mathbf{w}}=\Pi_1\mathbf{w}$, show that

$$\widetilde{\mathbf{w}}=L(L^{\mathsf{T}}L)^{-1}U^{-\mathsf{T}}\widetilde{\mathbf{c}}$$

where $U^{-\mathsf{T}}=(U^{-1})^{\mathsf{T}}=(U^{\mathsf{T}})^{-1}$, a standard notation that we will also use below. (*Hint*: You'd need to use something that you've already determined in an earlier part).

(b) Write

$$\widetilde{\mathbf{w}}=\begin{bmatrix}\widetilde{\mathbf{w}}_1\\ \widetilde{\mathbf{w}}_2\end{bmatrix}$$

where $\widetilde{\mathbf{w}}_1\in\mathbb{R}^n$ and $\widetilde{\mathbf{w}}_2\in\mathbb{R}^{m-n}$. Show that

$$\widetilde{\mathbf{w}}_1=L_1^{-\mathsf{T}}U^{-\mathsf{T}}\widetilde{\mathbf{c}}-S^{\mathsf{T}}\widetilde{\mathbf{w}}_2.$$

(c) Write $\mathbf{d}=L_1^{-\mathsf{T}}U^{-\mathsf{T}}\widetilde{\mathbf{c}}$. Deduce that $\widetilde{\mathbf{w}}_2$ may be obtained either as a solution to

$$\min_{\widetilde{\mathbf{w}}_2\in\mathbb{R}^{m-n}}\left\|\begin{bmatrix}S^{\mathsf{T}}\\ I_{m-n}\end{bmatrix}\widetilde{\mathbf{w}}_2-\begin{bmatrix}\mathbf{d}\\ \mathbf{0}\end{bmatrix}\right\|_2$$

or as

$$\widetilde{\mathbf{w}}_2=(I_{m-n}+SS^{\mathsf{T}})^{-1}S\mathbf{d}.$$

Note that when $m-n<n$, this method is advantageous for the same reason in Problem **3**.

5. So far we have assumed that A has full column rank. Suppose now that $\operatorname{rank}(A)=r<\min\{m,n\}$.

(a) Show that the LU factorization obtained using GECP is of the form

$$\Pi_1A\Pi_2=LU=\begin{bmatrix}L_1\\ L_2\end{bmatrix}\begin{bmatrix}U_1 & U_2\end{bmatrix}$$

where $L_1,U_1\in\mathbb{R}^{r\times r}$ are triangular and nonsingular.

(b) Show that the above equation may be rewritten in the form

$$\Pi_1 A \Pi_2 = \begin{bmatrix} I_r \\ S_1 \end{bmatrix} L_1 U_1 \begin{bmatrix} I_r & S_2^{\mathsf{T}} \end{bmatrix}$$

for some matrices S_1 and S_2.

(c) Hence show that the Moore–Penrose inverse of A is given by

$$A^{\dagger} = \Pi_2 \begin{bmatrix} I_r & S_2^{\mathsf{T}} \end{bmatrix}^{\dagger} U_1^{-1} L_1^{-1} \begin{bmatrix} I_r \\ S_1 \end{bmatrix}^{\dagger} \Pi_1.$$

(d) Using the general formula (derived in the lectures) for the Moore–Penrose inverse of a rank-retaining factorization, what do you get for $A^{\dagger}$?

6. Consider the block matrix

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{R}^{n \times n}$$

where $A \in \mathbb{R}^{p \times p}$, $B \in \mathbb{R}^{p \times q}$, $C \in \mathbb{R}^{q \times p}$, $D \in \mathbb{R}^{q \times q}$ and $n = p + q$. The Schur complements of A and D are

$$S = D - CA^{\dagger}B \quad \text{and} \quad T = A - BD^{\dagger}C$$

respectively.

(a) Verify that if A and S are nonsingular, then

$$X^{-1} = \begin{bmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{bmatrix}$$

and if D and T are nonsingular, then

$$X^{-1} = \begin{bmatrix} T^{-1} & -T^{-1}BD^{-1} \\ -D^{-1}CT^{-1} & D^{-1} + D^{-1}CT^{-1}BD^{-1} \end{bmatrix}.$$

(b) Show that

$$\det X = \begin{cases} \det(A)\det(D - CA^{-1}B) & \text{if } A \text{ nonsingular,} \\ \det(D)\det(A - BD^{-1}C) & \text{if } D \text{ nonsingular.} \end{cases}$$

Deduce that

$$\det(A + BC) = \det(A)\det(I + CA^{-1}B)$$

and use it to find the determinants of the following matrices

$$\begin{bmatrix} \frac{1+\lambda_1}{\lambda_1} & 1 & \cdots & 1 \\ 1 & \frac{1+\lambda_2}{\lambda_2} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & \frac{1+\lambda_n}{\lambda_n} \end{bmatrix}, \quad \begin{bmatrix} 1+\lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1 & 1+\lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 & \lambda_2 & \cdots & 1+\lambda_n \end{bmatrix}, \quad \begin{bmatrix} \lambda & \mu & \mu & \cdots & \mu \\ \mu & \lambda & \mu & \cdots & \mu \\ \mu & \mu & \lambda & \cdots & \mu \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu & \mu & \mu & \cdots & \lambda \end{bmatrix}.$$

(c) Show that if A has all principal matrices nonsingular so that we may perform Gaussian elimination without pivoting to A, then applying the first p steps of that to X yields

$$X = \begin{bmatrix} L_{11} & 0 \\ L_{21} & I_q \end{bmatrix} \begin{bmatrix} I_p & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ 0 & I_q \end{bmatrix}$$

where $A = L_{11}U_{11}$ is the LU factorization of A. What are L_{21} and U_{12} in terms of L_{11}, U_{11} and the blocks of X?

(d) Suppose X is symmetric (so $C = B^\intercal$) and A is positive definite. Show that applying the first p steps of Cholesky factorization to X yields

$$X = \begin{bmatrix} R_{11}^\intercal \\ R_{12}^\intercal \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix}$$

where $A = R_{11}^\intercal R_{11}$ is the Cholesky factorization. What is R_{12} in terms of R_{11} and the blocks of X?