STAT 309: MATHEMATICAL COMPUTATIONS I
FALL 2021
PROBLEM SET 0

This homework mostly serves as a linear algebra refresher. We will recall some definitions. The null space or kernel of a matrix $A \in \mathbb{R}^{m\times n}$ is the set

$$\ker(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$$

while the range space or image is the set

$$\operatorname{im}(A) = \{\mathbf{y} \in \mathbb{R}^m : \mathbf{y} = A\mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{R}^n\}.$$

The rank and nullity of A are defined as the dimensions of these spaces,

$$\operatorname{rank}(A) = \dim \operatorname{im}(A) \qquad \text{and} \qquad \operatorname{nullity}(A) = \dim \ker(A).$$

By convention we write all vectors in $\mathbb{R}^n$ as column vectors.

1. (a) For $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$, show that

$$\operatorname{im}(AB) \subseteq \operatorname{im}(A) \qquad \text{and} \qquad \ker(AB) \supseteq \ker(B).$$

When does equality occur in each of these inclusions?

(b) For $A, B \in \mathbb{R}^{n\times n}$, show that

$$\operatorname{rank}(AB) \le \min\{\operatorname{rank}(A), \operatorname{rank}(B)\},$$
$$\operatorname{nullity}(AB) \le \operatorname{nullity}(A) + \operatorname{nullity}(B),$$
$$\operatorname{rank}(A + B) \le \operatorname{rank}(A) + \operatorname{rank}(B).$$

(c) For $A, B \in \mathbb{R}^{n\times n}$, show that if $AB = 0$, then

$$\operatorname{rank}(A) + \operatorname{rank}(B) \le n.$$

2. (a) Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$. Show that

$$\operatorname{rank}\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = \operatorname{rank}(A) + \operatorname{rank}(B).$$

We have used the block matrix notation here. For example if $A = \left[\begin{smallmatrix} a & b & c \\ d & e & f \end{smallmatrix}\right] \in \mathbb{R}^{2\times 3}$ and $B = \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right] \in \mathbb{R}^{2\times 1}$, then

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} a & b & c & 0 \\ d & e & f & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & \beta \end{bmatrix} \in \mathbb{R}^{4\times 4}.$$

This is sometimes also denoted as $A \oplus B$. It is a direct sum of operators induced by a direct sum of vector spaces.

(b) For $\mathbf{x} = [x_1, \dots, x_m]^\mathsf{T} \in \mathbb{R}^m$ and $\mathbf{y} = [y_1, \dots, y_n]^\mathsf{T} \in \mathbb{R}^n$, observe that $\mathbf{x}\mathbf{y}^\mathsf{T} \in \mathbb{R}^{m\times n}$. Let $A \in \mathbb{R}^{m\times n}$. Show that $\operatorname{rank}(A) = 1$ iff $A = \mathbf{x}\mathbf{y}^\mathsf{T}$ for some nonzero $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{y} \in \mathbb{R}^n$.

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3. Let $A \in \mathbb{R}^{m\times n}$.

(a) Show that
$$\ker(A^\mathsf{T}A) = \ker(A) \qquad \text{and} \qquad \operatorname{im}(A^\mathsf{T}A) = \operatorname{im}(A^\mathsf{T}).$$
Give an example to show this is not true over a finite field (e.g. a field of two elements $\mathbb{F}_2 = \{0, 1\}$ with binary arithmetic).

(b) Show that
$$A^\mathsf{T}A\mathbf{x} = A^\mathsf{T}\mathbf{b}$$
always has a solution (even if $A\mathbf{x} = \mathbf{b}$ has no solution). Give an example to show that this is not true over a finite field.

4. Let $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{R}^n$. Let $r \le n$ and $G_r = [g_{ij}] \in \mathbb{R}^{r\times r}$ be the matrix with
$$g_{ij} = \mathbf{v}_i^\mathsf{T}\mathbf{v}_j$$
for $i, j = 1, \dots, r$. This is called a *Gram matrix.*

(a) Show that $\mathbf{v}_1, \dots, \mathbf{v}_r$ are linearly independent iff $\operatorname{nullity}(G_r) = 0$.

(b) Show that $G_r = I_r$ iff $\mathbf{v}_1, \dots, \mathbf{v}_r$ are pairwise orthogonal unit vectors, i.e., $\|\mathbf{v}_i\|_2 = 1$ for all $i = 1, \dots, r$, and $\mathbf{v}_i^\mathsf{T}\mathbf{v}_j = 0$ for all $i \ne j$. If this holds, show that
$$\sum_{i=1}^{r} (\mathbf{v}^\mathsf{T}\mathbf{v}_i)^2 \le \|\mathbf{v}\|_2^2 \tag{4.1}$$
for all $\mathbf{v} \in \mathbb{R}^n$. What can you say about $\mathbf{v}_1, \dots, \mathbf{v}_r$ if equality always holds in (**??**) for all $\mathbf{v} \in \mathbb{R}^n$?

5. Let $A \in \mathbb{C}^{n\times n}$. Recall that A is diagonalizable iff there exists an invertible $X \in \mathbb{C}^{n\times n}$ such that $X^{-1}AX = \Lambda$, a diagonal matrix.

(a) Show that A is diagonalizable if and only if its minimal polynomial is of the form
$$m_A(x) = (x - \lambda_1) \cdots (x - \lambda_d)$$
where $\lambda_1, \dots, \lambda_d \in \mathbb{C}$ are all distinct. Hence deduce for a diagonalizable matrix, the degree of its minimal polynomial equals the number of distinct eigenvalues.

(b) Let A be diagonalizable. Let $\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{C}^n$ be n linearly independent right eigenvectors, i.e., $A\mathbf{x}_i = \lambda_i\mathbf{x}_i$; and $\mathbf{y}_1, \dots, \mathbf{y}_n \in \mathbb{C}^n$ be n linearly independent left eigenvectors, i.e., $\mathbf{y}_i^\mathsf{T}A = \lambda_i\mathbf{y}_i^\mathsf{T}$. Show that there is a choice of left and right eigenvectors of A such that any vector $\mathbf{v} \in \mathbb{C}^n$ can be expressed as
$$\mathbf{v} = \sum_{i=1}^{n} (\mathbf{y}_i^\mathsf{T}\mathbf{v})\mathbf{x}_i.$$
If we write $X = [\mathbf{x}_1, \dots, \mathbf{x}_n] \in \mathbb{C}^{n\times n}$ and $Y = [\mathbf{y}_1, \dots, \mathbf{y}_n] \in \mathbb{C}^{n\times n}$. What is the relation between X and Y?