1. Let $A \in \mathbb{C}^{n \times n}$ be nonsingular. We would like to solve $Ax = b$ with the splitting $A = M - N$ where $M$ is nonsingular. Let $B = M^{-1}N$ and $c = M^{-1}b$. Consider the iteration
\[ x^{(k+1)} = Bx^{(k)} + c. \quad (1.1) \]
By applying Problem Set 1, Problem 4(d) or otherwise, show that (1.1) converges to the solution of $Ax = b$ for all $x^{(0)}$ and all $b$ if and only if $\rho(B) < 1$.

2. In general, a semi-iterative method is one that comprises two steps:
\[ x^{(k+1)} = Mx^{(k)} + b \]  
(Iteration)
and
\[ y^{(m)} = \sum_{k=0}^{m} \alpha^{(m)}_k x^{(k)}. \]  
(Extrapolation)
As in the lectures, we will assume that $M = I - A$ with $\rho(M) < 1$ and that we are interested to solve $Ax = b$ for some nonsingular matrix $A \in \mathbb{C}^{n \times n}$. Let
\[ e^{(k)} = x^{(k)} - x \quad \text{and} \quad \epsilon^{(m)} = y^{(m)} - x. \]
(a) By considering what happens when $x^{(0)} = x$, show that it is natural to impose
\[ \sum_{k=0}^{m} \alpha^{(m)}_k = 1 \quad (2.2) \]
for all $m \in \mathbb{N} \cup \{0\}$. Henceforth, we will assume that (2.2) is satisfied for all problems in this problem set.
(b) Show that for all $m \in \mathbb{N}$, we may write
\[ \epsilon^{(m)} = P_m(M)e^{(0)} \]
for some $P_m(x) = \alpha^{(m)}_0 + \alpha^{(m)}_1 x + \cdots + \alpha^{(m)}_m x^m \in \mathbb{C}[x]$ with $\deg(P_m) \leq m$ and $P_m(1) = 1$.
(c) Hence deduce that a necessary condition for $\epsilon^{(m)} \to 0$ is that
\[ \lim_{m \to \infty} \|P_m(M)\|_2 < 1 \]
where $\|\cdot\|_2$ is the spectral norm. Is this condition also sufficient?
(d) Consider the case when
\[ \alpha^{(m)}_0 = \alpha^{(m)}_1 = \cdots = \alpha^{(m)}_m = \frac{1}{m+1} \]
for all $m \in \mathbb{N} \cup \{0\}$. Show that if a sequence (any sequence, not necessarily one generated as in (Iteration)) is convergent and
\[ \lim_{k \to \infty} x^{(k)} = x \]
then
\[ \lim_{m \to \infty} y^{(m)} = x. \]

Is the converse also true?

3. It is clear that in any semi-iterative method defined by some \( M \in \mathbb{C}^{n \times n} \) with \( \rho(M) < 1 \), we would like to solve the problem
\[
\min_{P \in \mathbb{C}[x], \deg(P)=m, P(1)=1} \|P(M)\|_2. \tag{3.3}
\]

Note that the condition \( P(1) = 1 \) is motivated by Problem 2(a).

(a) Show that if \( m \geq n \), then a solution to (3.3) is given by
\[
P_m(x) = \frac{x^{m-n} \det(xI - M)}{\det(I - M)}.
\]

You may assume the Cayley–Hamilton Theorem. How do we know that the denominator is non-zero?

(b) From now on assume that \( M \) is Hermitian with minimum and maximum eigenvalues \( \lambda_{\min} := a \) and \( \lambda_{\max} := b \in \mathbb{R} \). Define
\[
\|f\|_\infty = \sup_{x \in [a,b]} |f(x)|.
\]

Emulating our discussions in the lectures, show that for \( m = 0, 1, \ldots, n-1 \), the solution to the relaxed problem
\[
\min_{P \in \mathbb{C}[x], \deg(P)=m, P(1)=1} \|P\|_\infty \tag{3.4}
\]
would yield an upper bound to (3.3).

(c) Consider the Chebyshev polynomials defined by
\[
C_m(x) = \begin{cases} 
\cos(m \cos^{-1}(x)) & -1 \leq x \leq 1, \\
\cosh(m \cosh^{-1}(x)) & x > 1, \\
(-1)^m \cosh(m \cosh^{-1}(-x)) & x < -1.
\end{cases}
\]

Suppose \(-1 < a < b < +1\). Show that the polynomials defined by
\[
P_m(x) = \frac{C_m \left( \frac{2x - (b + a)}{b - a} \right)}{C_m \left( \frac{2 - (b + a)}{b - a} \right)} \tag{3.5}
\]
satisfy \( \deg(P_m) = m \), \( P_m(1) = 1 \), and
\[
\|P_m\|_\infty = \frac{1}{C_m \left( \frac{2 - (b + a)}{b - a} \right)}.
\]

(d) By emulating our discussions in the lectures, show that the solution to (3.4) is given by \( P_m \). Note that this solves (3.4) for all \( m \in \mathbb{N} \) and not just \( m \leq n - 1 \).

(e) Show that the solution in (d) is unique.

4. Let \( M \in \mathbb{C}^{n \times n} \) be Hermitian with \( \rho(M) = \rho < 1 \). Moreover, suppose that
\[
\lambda_{\min} = -\rho, \quad \lambda_{\max} = \rho.
\]
(a) Show that the $P_m$'s in (3.5) satisfy a three-term recurrence relation
\[ C_{m+1} \left( \frac{1}{\rho} \right) P_{m+1}(x) = \frac{2x}{\rho} C_m \left( \frac{1}{\rho} \right) P_m(x) - C_{m-1} \left( \frac{1}{\rho} \right) P_{m-1}(x) \]
for all $m \in \mathbb{N}$.

(b) Show that the semi-iterative method with $\alpha_k^{(m)}$ given by the coefficient of $P_m$ in (3.5) may be written as
\[ y^{(m+1)} = \omega_{m+1} (My^{(m)} - y^{(m-1)} + b) + y^{(m-1)} \]
where $y^{(-1)} := 0$, $\omega_1 := 1$, and
\[ \omega_{m+1} = \frac{2C_m(1/\rho)}{\rho C_{m+1}(1/\rho)} \]
for $m = 0, 1, 2, \ldots$. This is a slightly different Chebyshev method where we choose the normalization (2.2) instead of $\alpha_m^{(m)} = 1$ in the lecture.

(c) Show that $\|P_m(M)\|_2 = \frac{1}{C_m(1/\rho)} = \frac{1}{\cosh(m\sigma)}$
where $\sigma = \cosh^{-1}(1/\rho)$. Deduce that $\|P_m(M)\|_2$ is a strictly decreasing sequence for all $m = 0, 1, 2 \ldots$.

(d) Show that
\[ e^{-\sigma} = (\omega - 1)^{1/2} \]
where
\[ \omega = \frac{2}{1 + \sqrt{1 - \rho^2}} \]
and deduce that
\[ \|P_m(M)\|_2 = \frac{2(\omega - 1)^{m/2}}{1 + (\omega - 1)^m}. \]

(e) Hence show that $(\omega_m)^{\infty}_{m=0}$ is strictly decreasing for $m \geq 2$ and that
\[ \lim_{m \to \infty} \omega_m = \omega. \]

5. Let $M \in \mathbb{C}^{n \times n}$ be nonsingular with $\rho(M) < 1$ and suppose we are interested in solving
\[ Mx = b. \]

(a) Show that SOR applied to the system
\[ \begin{bmatrix} I & -M \\ -M & I \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} b \\ b \end{bmatrix} \]
yields the following iterations
\[ x^{(m+1)} = \omega(Mx^{(m)} - x^{(m)} + b) + x^{(m)}, \]
\[ z^{(m+1)} = \omega(Mz^{(m+1)} - z^{(m)} + b) + z^{(m)}, \]
for $m = 0, 1, 2, \ldots$.

(b) Define the sequence of iterates $y^{(m)}$ by
\[ y^{(m)} = \begin{cases} x^{(k)} & \text{if } m = 2k, \\ z^{(k)} & \text{if } m = 2k + 1. \end{cases} \]
Show that the iterations obtained in (a) are exactly the iterations in Problem 4(b). This shows that SOR applied to (5.8) is equivalent to Chebyshev applied to (5.7) but with $\omega_m = \omega$. 

for all \( m \in \mathbb{N} \). Note that if \( \omega \) is chosen to be the value in (4.6), then this is in fact the optimal SOR parameter.

6. Let \( A \in \mathbb{R}^{n \times n} \) be symmetric positive definite and \( b \in \mathbb{R}^n \). As usual, we write
\[
 r_k = b - Ax_k. \tag{6.9} 
\]
We assume that \( x_0 \) is initialized in some manner. In the lectures we assumed \( x_0 = 0 \) and so \( r_0 = b \) but we will do it a little more generally here. Consider the quadratic functional
\[
 \varphi(x) = x^T A x - 2 b^T x. 
\]
(a) Show that
\[
 \nabla \varphi(x_k) = -2r_k 
\]
and hence if \( x_* \in \mathbb{R}^n \) is a stationary point of \( \varphi \), then
\[
 Ax_* = b. 
\]
Show also that \( x_* \) must be a minimizer of \( \varphi \).

(b) Consider an iterative method
\[
 x_{k+1} = x_k + \alpha_k p_k \tag{6.10} 
\]
where \( p_0, p_1, p_2, \ldots \) are search directions to be chosen later. Show that if we want \( \alpha_k \) so that the function \( f : \mathbb{R} \to \mathbb{R} \)
\[
 f(\alpha) = \varphi(x_k + \alpha p_k) 
\]
is minimized, then we must have
\[
 \alpha_k = \frac{r_k^T p_k}{p_k^T A p_k}. \tag{6.11} 
\]
(c) Deduce that
\[
 \varphi(x_{k+1}) - \varphi(x_k) = -\frac{(r_k^T p_k)^2}{p_k^T A p_k} 
\]
and therefore \( \varphi(x_{k+1}) < \varphi(x_k) \) as long as \( r_k^T p_k \neq 0 \).

(d) Show that if we choose
\[
 p_k = r_k, \tag{6.12} 
\]
we obtain the steepest decent method discussed in the lectures.
(e) Let the eigenvalues of \( A \) be \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0 \) and \( P \in \mathbb{R}[t] \). Show that
\[
 \| P(A) x \|_A \leq \max_{1 \leq i \leq n} |P(\lambda_i)| \| x \|_A 
\]
for every \( x \in \mathbb{R}^n \). [Hint: \( A > 0 \) and so has an eigenbasis].
(f) Using (e) and \( P_\alpha(t) = 1 - \alpha t \), show that if we have (6.12), then
\[
 \| x_k - x_* \|_A \leq \max_{1 \leq i \leq n} |P_\alpha(\lambda_i)| \| x_{k-1} - x_* \|_A 
\]
for all \( \alpha \in \mathbb{R} \).

(g) Using properties of Chebyshev polynomials, show that
\[
 \min_{\alpha \in \mathbb{R}} \max_{\lambda_\alpha \leq t \leq \lambda_1} |1 - \alpha t| = \frac{\lambda_1 - \lambda_\alpha}{\lambda_1 + \lambda_\alpha} 
\]
and hence deduce that
\[
 \| x_k - x_* \|_A \leq \frac{\lambda_1 - \lambda_\alpha}{\lambda_1 + \lambda_\alpha} \| x_{k-1} - x_* \|_A. 
\]