1. Let $x \in \mathbb{C}^n$ and $A \in \mathbb{C}^{m \times n}$. We write $\|x\|_2 = \sqrt{x^*x}$ and $\|A\|_2 = \sup_{\|x\|_2 = 1} \|Ax\|_2$ for the vector 2-norm and matrix 2-norm respectively.
   (a) Show that there is no ambiguity in the notation, i.e., if $A \in \mathbb{C}^{n \times 1}$, then $\|A\|_2$ is the same whether we regard it as the vector or matrix 2-norm. What if $A \in \mathbb{C}^{1 \times n}$?
   (b) Show that the vector 2-norm is unitarily invariant, i.e.,
   $$\|UX\|_2 = \|x\|_2$$
   for all unitary matrices $U \in \mathbb{C}^{n \times n}$.
   (c) Bonus: Show that no other vector $p$-norm is unitarily invariant, $1 \leq p \leq \infty$, $p \neq 2$.
   (d) Show that the matrix 2-norm is unitarily invariant, i.e.,
   $$\|UAV\|_2 = \|A\|_2$$
   for all unitary matrices $U \in \mathbb{C}^{m \times m}$, $V \in \mathbb{C}^{n \times n}$.
   (e) Show that the Frobenius norm is unitarily invariant, i.e.,
   $$\|UAV\|_F = \|A\|_F$$
   for all unitary matrices $U \in \mathbb{C}^{m \times m}$, $V \in \mathbb{C}^{n \times n}$. (Hint: First show that $\|A\|_F^2 = \text{tr}(A^*A) = \text{tr}(AA^*)$).
   (f) Let $U \in \mathbb{C}^{n \times n}$. Show that the following are equivalent statements:
   (i) $\|UX\|_2 = \|x\|_2$ for all $x \in \mathbb{C}^n$;
   (ii) $(UX)^*UY = x^*y$ for all $x, y \in \mathbb{C}^n$;
   (iii) $U$ is unitary.

2. Let $A \in \mathbb{C}^{n \times n}$. Let $\|\cdot\|$ be an operator norm of the form
   $$\|A\| = \max_{0 \neq v \in \mathbb{C}^n} \frac{\|Av\|_\alpha}{\|v\|_\alpha}$$
   for some vector norm $\|\cdot\|_\alpha : \mathbb{C}^n \to [0, \infty)$. Show that if $\|A\| < 1$, then $I - A$ is nonsingular and furthermore,
   $$\frac{1}{1 + \|A\|} \leq \|(I - A)^{-1}\| \leq \frac{1}{1 - \|A\|}.$$

3. We will examine the effect of various parameters on the accuracy of a computed solution to a nonsingular linear system. Relevant commands in Matlab syntax are given in brackets.
   (a) Generate $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ as follows:
      (i) $a_{ij}$ randomly generated from a standard normal distribution $\text{randn}(n)$;
      (ii) a Hilbert matrix, i.e., $a_{ij} = 1/(i + j - 1)$ $\text{hilb}(n)$;
      (iii) a Pascal matrix, i.e., the entries $a_{ij} = \binom{i+j}{i}$ $\text{pascal}(n)$;

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(iv) a magic square, i.e., the entries \( a_{ij} \)'s are the integers \( 1, 2, \ldots, n^2 \) arranged in a way that \( A \) has equal row, column, and diagonal sums [\text{magic}(n)].

\[
\text{hilb}(4) = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/3 & 1/4 & 1/5 \\ 1/3 & 1/4 & 1/5 & 1/6 \\ 1/4 & 1/5 & 1/6 & 1/7 \end{bmatrix}, \quad \text{pascal}(4) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}, \quad \text{magic}(4) = \begin{bmatrix} 16 & 2 & 3 & 13 \\ 5 & 11 & 10 & 8 \\ 9 & 7 & 6 & 12 \\ 4 & 14 & 15 & 1 \end{bmatrix}
\]

For simplicity, we will assume that \( A \) is stored exactly with no errors even though this is only true for those matrices with integer-valued entries.

(b) Generate \( x \) and \( b \in \mathbb{R}^n \) as follows:
   (i) \( x = [1, \ldots, 1]^T \cdot [\text{ones}(n,1)] \);
   (ii) \( b = Ax \cdot [b = A\cdot x] \).

(c) For each \( A \) generated as above, perform the following for \( n = 5, 10, 15, \ldots, 500 \).
   (i) Solve \( Ax = b \) using your program to get \( \hat{x} \cdot [\text{hat} = A\cdot b] \).
      Note that in general the result computed by your program will not be exactly the true solution \( x = A^{-1}b \) because of roundoff errors that occurred during computations.
   (ii) Compute \( \delta b = A\hat{x} - b \) and record the values of \( \|x - \hat{x}||/\|x\|, \kappa(A) = \|A\||\|A^{-1}\| \) and \( \kappa(A)\|\delta b\|/\|b\| \) for \( \|\cdot\| = \|\cdot\|_1, \|\cdot\|_2 \), and \( \|\cdot\|_\infty \).
   (iii) Present everything for the \( n = 5 \) case but only tabulate the relevant trend for general \( n > 5 \) in a graph.

(d) Discuss and explain the effects of different choices of \( A, b, \|\cdot\|, \) and \( n \) have on the accuracy of the computed solution \( \hat{x} \).

(e) Instead of solving the linear system directly, compute \( A^{-1} \) and then \( \hat{x} := A^{-1}b \cdot [\text{hat} = \text{inv}(A)\cdot b] \).
   Comment on the accuracy of this approach. Provide numerical evidence to support your conclusion.

(f) Write a program that computes the \( (1,1) \)-entry of the matrix \( A^{-1} \) that does not involve computing \( A^{-1} \) i.e., if \( A^{-1} = [b_{ij}] \), you want the value \( b_{11} \) but you are not allowed to compute \( A^{-1} \).

4. Let \( A \in \mathbb{R}^{n \times n} \) be nonsingular and let \( 0 \neq b \in \mathbb{R}^n \). Let \( x = A^{-1}b \in \mathbb{R}^n \). In the following, \( \delta A \in \mathbb{R}^{n \times n} \) and \( \delta b \in \mathbb{R}^n \) are some arbitrary matrix and vector. We assume that the norm on \( A \) satisfies \( \|Ax\| \leq \|A\|\|x\| \) for all \( A \in \mathbb{R}^{n \times n} \) and all \( x \in \mathbb{R}^n \).

\[(a) \text{ Show that if } \delta A \in \mathbb{R}^{n \times n} \text{ is any matrix satisfying }
\frac{\|\delta A\|}{\|A\|} \leq \frac{1}{\kappa(A)}, \quad (4.2)\]
then \( A + \delta A \) must be nonsingular. (\textit{Hint:} If \( A + \delta A \) is singular, then there exists nonzero \( v \) such that \( (A + \delta A)v = 0)\).

\[(b) \text{ Suppose } (A + \delta A)(x + \delta x) = b \text{ and } \hat{x} = x + \delta x. \text{ Show that }
\frac{\|\delta x\|}{\|\hat{x}\|} \leq \kappa(A) \frac{\|\delta A\|}{\|A\|}. \quad (4.3)\]

\[(c) \text{ Suppose } (A + \delta A)(x + \delta x) = b \text{ and } \hat{x} = x + \delta x \text{ and }(4.2)\text{ is satisfied. Show that }
\frac{\|\delta x\|}{\|x\|} \leq \frac{\kappa(A)\|\delta A\|}{1 - \kappa(A)\|\delta A\|}.\]

You may like use the following outline:
   (i) Show that \( \delta x = -A^{-1}\delta Ax \)
and so
\[ \|\delta x\| \leq \kappa(A) \|\delta A\| (\|x\| + \|\delta x\|). \]

(ii) Rewrite this inequality as
\[ \left(1 - \kappa(A) \frac{\|\delta A\|}{\|A\|}\right) \|\delta x\| \leq \kappa(A) \frac{\|\delta A\|}{\|A\|} \|x\| \]
and use (4.2).

(d) **Bonus**: Suppose \((A + \delta A)\hat{x} = b + \delta b\) where \(\hat{b} = b + \delta b \neq 0\) and \(\hat{x} = x + \delta x \neq 0\). Show that
\[ \frac{\|\delta x\|}{\|\hat{x}\|} \leq \kappa(A) \left(\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta b\|}{\|b\|} + \frac{\|\delta A\| \|\delta b\|}{\|A\| \|\hat{x}\|}\right). \] (4.4)

You may like use the following outline:

(i) Show that
\[ \delta x = A^{-1}(\delta b - \delta A\hat{x}) \]
and so
\[ \frac{\|\delta x\|}{\|\hat{x}\|} \leq \kappa(A) \left(\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta b\|}{\|b\|}\right). \] (4.5)

(ii) Show that
\[ \frac{1}{\|\hat{x}\|} \leq \frac{\|A\| + \|\delta A\|}{\|b\|}. \] (4.6)

(iii) Combine (4.5) and (2.1) to get (4.4).

(e) **Bonus**: Suppose \((A + \delta A)\hat{x} = b + \delta b\) where \(\hat{b} = b + \delta b \neq 0\) and \(\hat{x} = x + \delta x \neq 0\) and (4.2) is satisfied. Use the same ideas in (b) to deduce that
\[ \frac{\|\delta x\|}{\|x\|} \leq \frac{\kappa(A) \left(\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta b\|}{\|b\|}\right)}{1 - \kappa(A) \frac{\|\delta A\|}{\|A\|}}. \]

5. Recall that in the lectures, we mentioned that (i) there are matrix norms that are not submultiplicative and an example is the H"older \(\infty\)-norm; (ii) we may always construct a norm that approximates the spectral radius of a given matrix \(A\) as closely as we want.

(a) Show that if \(\|\cdot\| : \mathbb{C}^{m \times n} \rightarrow \mathbb{R}\) is a norm, then there always exists a \(c > 0\) such that the constant multiple \(\|\cdot\|_c := c\|\cdot\|\) defines a submultiplicative norm, i.e.,
\[ \|AB\|_c \leq \|A\|_c \|B\|_c \]
for any \(A \in \mathbb{C}^{m \times n}\) and \(B \in \mathbb{C}^{n \times p}\) (even if \(\|\cdot\|\) does not have this property). Find the constant \(c\) for the H"older \(\infty\)-norm.

(b) Let \(J \in \mathbb{C}^{n \times n}\) be in Jordan form, i.e.,
\[ J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{bmatrix} \]
where each block \(J_r\), for \(r = 1, \ldots, k\), has the form
\[ J_r = \begin{bmatrix} \lambda_r & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_r \end{bmatrix}. \]
Let $\varepsilon > 0$ and $D_\varepsilon = \text{diag}(1, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{n-1})$. Verify that

$$D_\varepsilon^{-1} J D_\varepsilon = \begin{bmatrix} J_{1, \varepsilon} & & \\ & \ddots & \\ & & J_{k, \varepsilon} \end{bmatrix}$$

where $J_{r, \varepsilon}$ is the matrix you obtain by replacing the 1’s on the superdiagonal of $J_r$ by $\varepsilon$’s,

$$J_{r, \varepsilon} = \begin{bmatrix} \lambda_r & \varepsilon \\ & \ddots & \ddots \\ & & \ddots & \varepsilon \\ & & & \lambda_r \end{bmatrix}$$

(c) Show that

$$\|D_\varepsilon^{-1} J D_\varepsilon\|_\infty \leq \rho(J) + \varepsilon.$$

(d) Hence, or otherwise, show that for any given $A \in \mathbb{C}^{n \times n}$ and $\varepsilon > 0$, there exists an operator norm $\|\cdot\|$ of the form (2.1) with the property that

$$\rho(A) \leq \|A\| \leq \rho(A) + \varepsilon.$$

(Hint: Transform $A$ into Jordan form).