1. Let $x \in \mathbb{C}^n$ and $A \in \mathbb{C}^{m \times n}$. We write $\|x\|_2 = \sqrt{x^*x}$ and $\|A\|_2 = \sup_{\|x\|_2 = 1} \|Ax\|_2$ for the vector 2-norm and matrix 2-norm respectively.
   (a) Show that there is no ambiguity in the notation, i.e., if $A \in \mathbb{C}^{n \times 1} = \mathbb{C}^n$, then $\|A\|_2$ is the same whether we regard it as the vector or matrix 2-norm. What if $A \in \mathbb{C}^{1 \times n}$?
   (b) Show that the vector 2-norm is unitarily invariant, i.e.,
   \[ \|Ux\|_2 = \|x\|_2 \]
   for all unitary matrices $U \in \mathbb{C}^{n \times n}$.
   (c) Bonus: Show that no other vector $p$-norm is unitarily invariant, $1 \leq p \leq \infty$, $p \neq 2$.
   (d) Show that the matrix 2-norm is unitarily invariant, i.e.,
   \[ \|UAV\|_2 = \|A\|_2 \]
   for all unitary matrices $U \in \mathbb{C}^{m \times m}$, $V \in \mathbb{C}^{n \times n}$.
   (e) Show that the Frobenius norm is unitarily invariant, i.e.,
   \[ \|UAV\|_F = \|A\|_F \]
   for all unitary matrices $U \in \mathbb{C}^{m \times m}$, $V \in \mathbb{C}^{n \times n}$. (Hint: First show that $\|A\|_F^2 = \text{tr}(A^*A) = \text{tr}((AA^*)^*)$).
   (f) Let $U \in \mathbb{C}^{n \times n}$. Show that the following are equivalent statements:
      (i) $\|Ux\|_2 = \|x\|_2$ for all $x \in \mathbb{C}^n$;
      (ii) $(Ux)^*Uy = x^*y$ for all $x, y \in \mathbb{C}^n$;
      (iii) $U$ is unitary.

2. Let $A \in \mathbb{C}^{n \times n}$. Let $\| \cdot \|$ be an operator norm of the form
   \[ \|A\| = \max_{0 \neq v \in \mathbb{C}^n} \frac{\|Av\|_\alpha}{\|v\|_\alpha} \quad (2.1) \]
   for some vector norm $\| \cdot \|_\alpha : \mathbb{C}^n \to [0, \infty)$. Show that if $\|A\| < 1$, then $I - A$ is nonsingular and furthermore,
   \[ \frac{1}{1 + \|A\|} \leq \|(I - A)^{-1}\| \leq \frac{1}{1 - \|A\|}. \]

3. We will examine the effect of various parameters on the accuracy of a computed solution to a nonsingular linear system. Relevant commands in Matlab syntax are given in brackets.
   (a) Generate $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ as follows:
      (i) $a_{ij}$ randomly generated from a standard normal distribution $[\text{randn}(n)]$;
      (ii) a Hilbert matrix, i.e., $a_{ij} = 1/(i + j - 1)$ $[\text{hilb}(n)]$;
      (iii) a Pascal matrix, i.e., the entries $a_{ij} = \binom{i+j}{i}$ $[\text{pascal}(n)]$;
4. Let $A$ have equal row, column, and diagonal sums $[\text{magic}(n)]$. (a) Show that if $A$ is stored exactly with no errors even though this is only true for those matrices with integer-valued entries.

(b) Generate $x$ and $b \in \mathbb{R}^n$ as follows:
   (i) $x = [1, \ldots, 1]^T \cdot [\text{ones}(n,1)]$;
   (ii) $b = Ax \cdot [b = A\cdot x]$.

(c) Suppose $A$ generated as above, perform the following for $n = 5, 10, 15, \ldots, 500$.
   (i) Solve $Ax = b$ using your program to get $\hat{x}$ $[\text{xhat} = A\backslash b]$. Note that in general the result computed by your program will not be exactly the true solution $x = A^{-1}b$ because of roundoff errors that occurred during computations.
   (ii) Compute $\delta b = A\hat{x} - b$ and record the values of $\|x - \hat{x}\|/\|x\|$, $\kappa(A) = \|A\|||A^{-1}\|$ and $\kappa(A)\|\delta b\|/\|b\|$ for $\|\cdot\| = \|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$.
   (iii) Present everything for the $n = 5$ case but only tabulate the relevant trend for general $n > 5$ in a graph.

(d) Discuss and explain the effects of different choices of $A, b, \|\cdot\|$, and $n$ have on the accuracy of the computed solution $\hat{x}$.

(e) Instead of solving the linear system directly, compute $A^{-1}$ and then $\hat{x} := A^{-1}b$ $[\text{xhat} = \text{inv}(A)\cdot b]$. Comment on the accuracy of this approach. Provide numerical evidence to support your conclusion.

(f) Write a program that computes the $(1, 1)$-entry of the matrix $A^{-1}$ that does not involve computing $A^{-1}$, i.e., if $A^{-1} = [b_{ij}]$, you want the value $b_{11}$ but you are not allowed to compute $A^{-1}$.

4. Let $A \in \mathbb{R}^{n \times n}$ be nonsingular and let $0 \neq b \in \mathbb{R}^n$. Let $x = A^{-1}b \in \mathbb{R}^n$. In the following, $\delta A \in \mathbb{R}^{n \times n}$ and $\delta b \in \mathbb{R}^n$ are some arbitrary matrix and vector. We assume that the norm on $A$ satisfies $\|Ax\| \leq \|A\|\|x\|$ for all $A \in \mathbb{R}^{n \times n}$ and all $x \in \mathbb{R}^n$.

(a) Show that if $\delta A \in \mathbb{R}^{n \times n}$ is any matrix satisfying

$$\frac{\|\delta A\|}{\|A\|} < \frac{1}{\kappa(A)},$$

then $A + \delta A$ must be nonsingular. (Hint: If $A + \delta A$ is singular, then there exists nonzero $v$ such that $(A + \delta A)v = 0$).

(b) Suppose $(A + \delta A)(x + \delta x) = b$ and $\hat{x} = x + \delta x$. Show that

$$\frac{\|\delta x\|}{\|\hat{x}\|} \leq \kappa(A) \frac{\|\delta A\|}{\|A\|}.$$  \hspace{1cm} (4.3)

(c) Suppose $(A + \delta A)(x + \delta x) = b$ and $\hat{x} = x + \delta x$ and (4.2) is satisfied. Show that

$$\frac{\|\delta x\|}{\|x\|} \leq \frac{\kappa(A) \frac{\|\delta A\|}{\|A\|}}{1 - \kappa(A) \frac{\|\delta A\|}{\|A\|}}.$$  

You may like use the following outline:

(i) Show that $\delta x = -A^{-1}\delta A\hat{x}$
and so
\[ \| \delta x \| \leq \kappa(A) \frac{\| \delta A \|}{\| A \|} (\| x \| + \| \delta x \|). \]

(ii) Rewrite this inequality as
\[ \left( 1 - \kappa(A) \frac{\| \delta A \|}{\| A \|} \right) \| \delta x \| \leq \kappa(A) \frac{\| \delta A \|}{\| A \|} \| x \| \]
and use (4.2).

(d) Bonus: Suppose \( (A + \delta A) \hat{x} = b + \delta b \) where \( \hat{b} = b + \delta b \neq 0 \) and \( \hat{x} = x + \delta x \neq 0 \). Show that
\[ \frac{\| \delta x \|}{\| \hat{x} \|} \leq \kappa(A) \left( \frac{\| \delta A \|}{\| A \|} + \frac{\| \delta b \|}{\| b \|} \right). \] \quad (4.4)

You may like use the following outline:
(i) Show that
\[ \delta x = A^{-1}(\delta b - \delta A \hat{x}) \]
and so
\[ \frac{\| \delta x \|}{\| \hat{x} \|} \leq \kappa(A) \left( \frac{\| \delta A \|}{\| A \|} + \frac{\| \delta b \|}{\| b \|} \right). \] \quad (4.5)

(ii) Show that
\[ \frac{1}{\| \hat{x} \|} \leq \frac{\| A \| + \| \delta A \|}{\| b \|}. \] \quad (4.6)

(iii) Combine (4.5) and (2.1) to get (4.4).

(e) Bonus: Suppose \( (A + \delta A) \hat{x} = b + \delta b \) where \( \hat{b} = b + \delta b \neq 0 \) and \( \hat{x} = x + \delta x \neq 0 \) and (4.2) is satisfied. Use the same ideas in (b) to deduce that
\[ \frac{\| \delta x \|}{\| x \|} \leq \frac{\kappa(A) \left( \frac{\| \delta A \|}{\| A \|} + \frac{\| \delta b \|}{\| b \|} \right)}{1 - \kappa(A) \frac{\| \delta A \|}{\| A \|}}. \]

5. Recall that in the lectures, we mentioned that (i) there are matrix norms that are not submultiplicative and an example is the Hölder \( \infty \)-norm; (ii) we may always construct a norm that approximates the spectral radius of a given matrix \( A \) as closely as we want.

(a) Show that if \( \| \cdot \| : \mathbb{C}^{m \times n} \to \mathbb{R} \) is a norm, then there always exists a \( c > 0 \) such that the constant multiple \( \| \cdot \|_c := c \| \cdot \| \) defines a submultiplicative norm, i.e.,
\[ \| AB \|_c \leq \| A \|_c \| B \|_c \]
for any \( A \in \mathbb{C}^{m \times n} \) and \( B \in \mathbb{C}^{n \times p} \) (even if \( \| \cdot \| \) does not have this property). Find the constant \( c \) for the Hölder \( \infty \)-norm.

(b) Let \( J \in \mathbb{C}^{n \times n} \) be in Jordan form, i.e.,
\[ J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{bmatrix} \]
where each block \( J_r \), for \( r = 1, \ldots, k \), has the form
\[ J_r = \begin{bmatrix} \lambda_r & 1 \\ & \ddots & \ddots \\ & & \ddots & 1 \\ & & & \lambda_r \end{bmatrix}. \]
Let \( \varepsilon > 0 \) and \( D_{\varepsilon} = \text{diag}(1, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{n-1}) \). Verify that

\[
D_{\varepsilon}^{-1} J D_{\varepsilon} = 
\begin{bmatrix}
J_{1,\varepsilon} & & \\
& \ddots & \\
& & J_{k,\varepsilon}
\end{bmatrix}
\]

where \( J_{r,\varepsilon} \) is the matrix you obtain by replacing the 1’s on the superdiagonal of \( J_r \) by \( \varepsilon \)'s,

\[
J_{r,\varepsilon} = 
\begin{bmatrix}
\lambda_r & \varepsilon & & & \\
& \ddots & \ddots & & \\
& & \ddots & \ddots & \\
& & & \varepsilon & \\
& & & & \lambda_r
\end{bmatrix}
\]

(c) Show that

\[
\|D_{\varepsilon}^{-1} J D_{\varepsilon}\|_{\infty} \leq \rho(J) + \varepsilon.
\]

(d) Hence, or otherwise, show that for any given \( A \in \mathbb{C}^{n \times n} \) and \( \varepsilon > 0 \), there exists an operator norm \( \| \cdot \| \) of the form (2.1) with the property that

\[
\rho(A) \leq \|A\| \leq \rho(A) + \varepsilon.
\]

(*Hint: Transform \( A \) into Jordan form.*)

6. Let \( A = [a_{ij}] \) be an \( n \times n \) matrix with entries

\[
a_{ij} = \begin{cases} 
  n + 1 - \max(i, j) & i \leq j + 1, \\
  0 & i > j + 1.
\end{cases}
\]

This is an example of an upper Hessenberg matrix: it is upper triangular except that the entries on the subdiagonal \( a_{i,i+1} \) may also be non-zero. For \( n = 12 \) and \( n = 25 \), do the following\(^1\):

(a) Compute \( \|A\|_\infty \) and \( \|A\|_1 \).
(b) Compute \( \rho(A) \) and \( \|A\|_2 \).
(c) Using Gerschgorin’s theorem, describe the domain that contains all of the eigenvalues.
(d) Compute all of the eigenvalues and singular values of \( A \). How many of the eigenvalues are real and how many are complex?

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\(^1\)You may use any built-in functions of your program.