1. Let \( u \in \mathbb{C}^n, u \neq 0 \). A Householder matrix \( H_u \in \mathbb{C}^{n \times n} \) is defined by
\[
H_u = I - \frac{2uu^*}{\|u\|^2}.
\]
(a) Show that \( H_u \) is both Hermitian and unitary.
(b) Show that for any \( \alpha \in \mathbb{C} \), \( \alpha \neq 0 \),
\[
H_{\alpha u} = H_u.
\]
In other words, \( H_u \) only depends on the ‘direction’ of \( u \) and not on its ‘magnitude’.
(c) In general, given a matrix \( M \in \mathbb{C}^{n \times n} \) and a vector \( x \in \mathbb{C}^n \), computing the matrix-vector product \( Mx \) requires \( n \) inner products — one for each row of \( M \) with \( x \). Show that \( H_u x \) can be computed using only two inner products.
(d) Given \( a, b \in \mathbb{C}^n \) where \( a \neq e^{i\theta}b \) for any \( \theta \in [0, 2\pi) \) and \( \|a\|_2 = \|b\|_2 \). Find \( u \in \mathbb{C}^n, u \neq 0 \) such that
\[
H_u a = b.
\]
(e) Show that \( u \) is an eigenvector of \( H_u \). What is the corresponding eigenvalue?
(f) Show that every \( v \in \text{span}\{u\}^\perp \) (cf. orthogonal complement in Homework 3) is an eigenvector of \( H_u \). What are the corresponding eigenvalues? What is \( \dim(\text{span}\{u\}^\perp) \)?
(g) Find the eigenvalue decomposition of \( H_u \), i.e. find a unitary matrix \( U \) and a diagonal matrix \( \Lambda \) such that
\[
H_u = U \Lambda U^*.
\]
(Hint: Gram-Schmidt algorithm).

2. Let \( A \in \mathbb{R}^{m \times n} \) and suppose its complete orthogonal decomposition is given by
\[
A = Q_1 \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} Q_2^\top,
\]
where \( Q_1 \) and \( Q_2 \) are orthogonal, and \( L \) is an nonsingular lower triangular matrix. Recall that \( X \in \mathbb{R}^{n \times m} \) is the unique pseudo-inverse of \( A \) if the following Moore-Penrose conditions hold:
(i) \( AXA = A \),
(ii) \( XAX = X \),
(iii) \( (AX)^\top = AX \),
(iv) \( (XA)^\top = XA \)
and in which case we write \( X = A^\dagger \).
(a) Let
\[
A^{-} = Q_2 \begin{bmatrix} L^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q_1^\top, \quad Y \neq 0.
\]
Which of the four conditions (i)–(iv) are satisfied?
(b) Prove that
\[ A^\dagger = Q_2 \begin{bmatrix} L^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q_1^\top \]
by letting
\[ A^\dagger = Q_2 \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} Q_1^\top \]
and by completing the following steps:
- Using (i), prove that \( X_{11} = L^{-1} \).
- Using the symmetry conditions (iii) and (iv), prove that \( X_{12} = 0 \) and \( X_{21} = 0 \).
- Using (ii), prove that \( X_{22} = 0 \).

3. Let \( A \in \mathbb{R}^{m \times n} \), \( b \in \mathbb{R}^m \), and \( c \in \mathbb{R}^n \). We are interested in the least squares problem
\[
\min_{x \in \mathbb{R}^n} \| Ax - b \|_2^2. \tag{3.1}
\]
(a) Show that \( x \) is a solution to (3.1) if and only if \( x \) is a solution to the augmented system
\[
\begin{bmatrix} I & A \\ A^\top & 0 \end{bmatrix} \begin{bmatrix} r \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}. \tag{3.2}
\]
(b) Show that the \((m + n) \times (m + n)\) matrix in (3.2) is nonsingular if and only if \( A \) has full column rank.
(c) Suppose \( A \) has full column rank and the QR decomposition of \( A \) is
\[ A = Q \begin{bmatrix} R \\ 0 \end{bmatrix}. \]
Show that the solution to the augmented system
\[
\begin{bmatrix} I & A \\ A^\top & 0 \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix}
\]
can be computed from
\[
\begin{aligned}
    z &= R^{-\top} c, \\
    d_1 &= Q^\top b,
\end{aligned}
\]
and
\[
\begin{aligned}
    x &= R^{-1}(d_1 - z), \\
    y &= Q \begin{bmatrix} z \\ d_2 \end{bmatrix}.
\end{aligned}
\]
(d) Hence deduce that if \( A \) has full column rank, then
\[ A^\dagger = R^{-1} Q_1^\top \]
where \( Q = [Q_1, Q_2] \) with \( Q_1 \in \mathbb{R}^{m \times n} \) and \( Q_2 \in \mathbb{R}^{m \times (m - n)} \). Check that this agrees with the general formula derived for a rank-retaining factorization \( A = GH \) in the lectures.

4. Let \( A \in \mathbb{R}^{m \times n} \). Suppose we apply QR with column pivoting to obtain the decomposition
\[ A = Q \begin{bmatrix} R & S \\ 0 & 0 \end{bmatrix} \Pi^\top \]
where \( Q \) is orthogonal and \( R \) is upper triangular and invertible. Let \( x_B \) be the basic solution, i.e.
\[
x_B = \Pi \begin{bmatrix} R^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q^\top b,
\]
and let \( \hat{x} = A^\dagger b \). Show that
\[
\frac{\|x_B - \hat{x}\|_2}{\|\hat{x}\|_2} \leq \|R^{-1}S\|_2.
\]
(Hint: If $\Pi^\top x = (u^\top, v^\top)^\top$ and $Q^\top b = (c^\top, d^\top)^\top$, consider the associated linearly constrained least-squares problem

$$\min \|u\|_2^2 + \|v\|_2^2 \quad \text{s.t.} \quad Ru + Sv = c$$

and write down the augmented system for the constrained problem.)

5. In Homework 3, Problem 4, we discussed solution of the data least squares problem, solving $Ax \approx b$ in a least squares sense when the error occurs only in $A$. In this problem, we examine what happens when $A \in \mathbb{R}^{n \times n}$ is a symmetric matrix, i.e. $A^\top = A$. In this case, it is natural to assume that the error $E \in \mathbb{R}^{n \times n}$ is also symmetric. Given a symmetric $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$. Let

$$r = b - Ax$$

where $0 \neq x \in \mathbb{R}^n$. Consider the QR decomposition

$$[x, r] = QR$$

and observe that if $Ex = r$, then

$$(Q^\top EQ)(Q^\top x) = Q^\top r.$$ 

Show how to compute a symmetric $E \in \mathbb{R}^{n \times n}$ so that it attains

$$\min_{(A+E)x = b} \|E\|_F.$$ 

6. In the following, $\kappa(A) := \|A\|\|A^\dagger\|$ for $A \in \mathbb{C}^{m \times n}$ where $\|\cdot\|$ denotes a submultiplicative matrix norm. We will write $\kappa_p(A)$ if the norm involved is a matrix $p$-norm.

(a) Show that for any $A \in \mathbb{C}^{m \times n}$,

$$\kappa(A) \geq 1.$$ 

(b) Show that for any $A \in \mathbb{C}^{m \times n}$,

$$\kappa_2(A^\ast A) = \kappa_2(A)^2$$

but that in general

$$\kappa(A^\ast A) \neq \kappa(A)^2.$$ 

(c) Show that for nonsingular $A, B \in \mathbb{C}^{n \times n}$,

$$\kappa(AB) \leq \kappa(A)\kappa(B).$$

Is this true in general without the nonsingular condition?

(d) Let $Q \in \mathbb{C}^{m \times n}$ be a matrix with orthonormal columns. Show that

$$\kappa_2(Q) = 1.$$ 

Is this true if $Q$ has orthonormal rows instead? Is this true with $\kappa_1$ or $\kappa_\infty$ in place of $\kappa_2$?

(e) Let $R \in \mathbb{C}^{n \times n}$ be a nonsingular upper-triangular matrix. Show that

$$\kappa_\infty(R) \geq \max_{i=1,\ldots,n} |r_{ii}| / \min_{i=1,\ldots,n} |r_{ii}|.$$ 

(f) Let $A \in \mathbb{R}^{m \times n}$. Show that

$$\min_{X \in \mathbb{R}^{n \times m}} \|AX - I_m\|_F$$

has a unique solution when $A$ has full column rank. In general, what is the minimum length solution, i.e. where $\|X\|_F$ is minimum?

(g) Let $b = [b_1, \ldots, b_n]^\top \in \mathbb{R}^n$ and $e = [1, \ldots, 1]^\top \in \mathbb{R}^n$. Solve

$$\min_{\beta \in \mathbb{R}} \|b - \beta e\|_p$$

for $p = 1, 2, \infty$. (Hint: The solutions $\beta_1, \beta_2, \beta_\infty$ are well-known notions in Statistics).