1. You are not allowed to use the SVD for this problem, i.e. no arguments should depend on the SVD of $A$ or $A^*$. Let $W$ be a subspace of $\mathbb{C}^n$. The subspace $W^\perp$ below is called the orthogonal complement of $W$.

$$ W^\perp = \{ v \in \mathbb{C}^n \mid v^* w = 0 \text{ for all } w \in W \}. $$

For any subspace $W \subseteq \mathbb{C}^n$, we write $P_W \in \mathbb{C}^{n \times n}$ for the projection onto $W$.

(a) Show that $\mathbb{C}^n = W \oplus W^\perp$ and that $W = (W^\perp)^\perp$.

(b) Let $A \in \mathbb{C}^{m \times n}$. Show that $\ker(A^*) = \im(A)^\perp$ and $\im(A^*) = \ker(A)^\perp$.

(c) Deduce the Fredholm alternative:

$$ \mathbb{C}^m = \ker(A^*) \oplus \im(A) \quad \text{and} \quad \mathbb{C}^n = \im(A^*) \oplus \ker(A). $$

In other words any $x \in \mathbb{C}^m$ and $y \in \mathbb{C}^n$ can be written uniquely as

$$ x = x_0 + x_1, \quad x_0 \in \ker(A), \quad x_1 \in \im(A^*), \quad x^*_0 x_1 = 0, $$

$$ y = y_0 + y_1, \quad y_0 \in \ker(A^*), \quad y_1 \in \im(A), \quad y^*_0 y_1 = 0. $$

(d) Show that

$$ x_0 = P_{\ker(A)} x, \quad x_1 = P_{\im(A^*)} x, \quad y_0 = P_{\ker(A^*)} y, \quad y_1 = P_{\im(A)} y. $$

(e) Consider the least squares problem for some $b \in \mathbb{C}^m$,

$$ \min_{x \in \mathbb{C}^n} \| b - A x \|_2. \quad (1.1) $$

Show that for any $x \in \mathbb{C}^n$,

$$ \| b - A x \|_2 \geq \| b_0 \|_2 $$

where $b_0 = P_{\ker(A^*)} b$. Deduce that $x \in \mathbb{C}^n$ is a solution to (1.1) if and only if

$$ A x = b_1 \quad \text{or, equivalently,} \quad b - A x = b_0. \quad (1.2) $$

Why is $A x = b_1$ consistent?

(f) Show that (1.2) is equivalent (i.e. if and only if) to the normal equation

$$ A^* A x = A^* b. \quad (1.3) $$

_Caveat_: In numerical analysis, it is an unforgivable sin to solve a least squares problem using its normal equation. Nonetheless (1.3) can be useful in mathematical arguments; just don’t ever use it for computations, instead use (1.2).

(g) Show that the pseudoinverse solution

$$ \min \left\{ \| x \|_2 : x \in \arg\min_{x \in \mathbb{C}^n} \| b - A x \|_2 \right\} $$

is given by

$$ x_1 = P_{\im(A^*)} x $$

where $x \in \mathbb{C}^n$ satisfies (1.2).
(h) Let $A \in \mathbb{C}^{n \times n}$ be normal, i.e. $A^*A = AA^*$. Show that
\[
\ker(A^*) = \ker(A) \quad \text{and} \quad \text{im}(A^*) = \text{im}(A)
\]
and deduce that for a normal matrix,
\[
\mathbb{C}^n = \ker(A) \oplus \text{im}(A).
\]

2. Let $A, B \in \mathbb{C}^{m \times n}$ with $n \leq m$. In the lectures, we claim that the solution $X \in U(n)$ to
\[
\min_{X^*X = I} \|A - BX\|_F
\]
is given by $X = UV^*$ where $B^*A = U\Sigma V^*$ is its singular value decomposition. Here we will prove it and consider some variants.

(a) Show that
\[
\|A - BX\|_F^2 = \text{tr}(A^*A) + \text{tr}(B^*B) - 2 \text{Re} \, \text{tr}(X^*B^*A)
\]
and deduce that the minimization problem is equivalent to
\[
\max_{X^*X = I} \text{Re} \, \text{tr}(X^*B^*A).
\]

(b) Show that
\[
\text{Re} \, \text{tr}(X^*B^*A) \leq \sum_{i=1}^n \sigma_i(B^*A)
\]
for any $X \in U(n)$. When is the upper bound attained?

(c) Show that
\[
\min_{X^*X = I} \|A - BX\|_F^2 = \sum_{i=1}^m (\sigma_i(A)^2 - 2\sigma_i(B^*A) + \sigma_i(B)^2).
\]

(d) Suppose $A$ has full column rank. Show that the following method produces a Hermitian matrix $X \in \mathbb{C}^{n \times n}$ that solves
\[
\min_{X^*X = X} \|AX - B\|_F. \tag{2.4}
\]

(i) Show that the SVD of $A$ takes the form
\[
A = U \begin{bmatrix} \Sigma \\ O \end{bmatrix} V^*
\]
where $U \in U(m), V \in U(n)$, and $\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_n) \in \mathbb{C}^{n \times n}$ is a diagonal matrix.

(ii) Show that
\[
\|AX - B\|_F^2 = \|\Sigma Y - C_1\|_F^2 + \|C_2\|_F^2
\]
where $Y = V^*XV$ and $C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = U^*BV$.

(iii) Note that $Y$ must be Hermitian if $X$ is. Show that
\[
\|\Sigma Y - C_1\|_F^2 = \sum_{i=1}^n |\sigma_i y_{ii} - c_{ii}|^2 + \sum_{j>i} |\sigma_i y_{ij} - c_{ij}|^2 + |\sigma_j y_{ij} - \bar{c}_{ji}|^2
\]
and deduce that the minimum value of (2.4) is attained when
\[
y_{ij} = \frac{\sigma_i c_{ij} + \sigma_j \bar{c}_{ji}}{\sigma_i^2 + \sigma_j^2}
\]
for all $i, j = 1, \ldots, n$. 

(e) Given $A \in \mathbb{C}^{n \times n}$. Describe how you would find $X \in \mathbb{C}^{n \times n}$ that solves
\[
\min_{\det(X) = \det(A)} \|A - X\|_F.
\]

(Hint: Consider the SVD of $A$).

3. Let $x \in \mathbb{C}^m$, $y \in \mathbb{C}^n$, and $A = xy^* \in \mathbb{C}^{m \times n}$.
(a) Show that $\text{rank}(A) = 1$ iff $x$ and $y$ are both non-zero. Such a matrix is usually called a rank-1 matrix.
(b) Show that
\[
\|A\|_F = \|A\|_2 = \|x\|_2\|y\|_2
\]
and that
\[
\|A\|_\infty \leq \|x\|_\infty\|y\|_1.
\]

What can you say about $\|A\|_1$?
(c) Let $x_1, \ldots, x_r \in \mathbb{C}^m$ be linearly independent and $y_1, \ldots, y_r \in \mathbb{C}^n$ be linearly independent. Let
\[
A = x_1y_1^* + \cdots + x_ry_r^*.
\]
Show that $\text{rank}(A) = r$. Show that this is not necessarily true if we drop either of the linear independence conditions.
(d) Given any $0 \neq A \in \mathbb{C}^{m \times n}$, show that
\[
\text{rank}(A) = \min\{r \in \mathbb{N} \mid A = \sum_{i=1}^r x_iy_i^*\}.
\]
In other words, the rank of a matrix is the smallest $r$ so that it may be expressed as a sum of $r$ rank-1 matrices.
(e) Show the following generalization of (3.5),
\[
\|A\|_F \leq \sqrt{\text{rank}(A)}\|A\|_2.
\]
Note that $\text{rank}_{cs}(A) = \|A\|_F^2/\|A\|_2^2$ is the ‘computer scientist’s numerical rank,’ one of the three notions of numerical ranks that we discussed. It is often used as a continuous surrogate for matrix rank.
(f) Show that with the nuclear norm we get instead
\[
\|A\|_* \leq \text{rank}(A)\|A\|_2.
\]
In other words we could also use $\|A\|_*/\|A\|_2$ as a continuous surrogate for matrix rank. In fact, this has been quite popular recently.

4. Let $A \in \mathbb{C}^{m \times n}$ and $b \in \mathbb{C}^m$. We will discuss a variant of $Ax \approx b$ where the error occurs only in $A$. Note that in ordinary least squares we assume that the error occurs only in $b$ while in total least squares we assume that it occurs in both $A$ and $b$.
(a) Show that if $0 \neq x \in \mathbb{C}^m$, then
\[
\left\| A \left( I - \frac{xx^*}{x^*x} \right) \right\|_F^2 = \|A\|_F^2 - \frac{\|Ax\|_2^2}{x^*x}.
\]
(b) Show that the matrix
\[
E = \frac{(b - Ax)x^*}{x^*x} \in \mathbb{C}^{m \times n}
\]
has the smallest 2-norm of all $m \times n$ matrices $E$ that satisfy
\[
(A + E)x = b.
\]
(c) What are the solutions of
\[
\min_{(A+E)x=b} \|E\|_2 \quad \text{and} \quad \min_{(A+E)x=b} \|E\|_F?
\]
(d) Given \(a \in \mathbb{C}^n\), \(b \in \mathbb{C}^m\), and \(\delta > 0\). Show how to solve the problems

\[
\min_{\|E\|_F \leq \delta} \|Ea - b\|_2 \quad \text{and} \quad \max_{\|E\|_F \leq \delta} \|Ea - b\|_2
\]

over all \(E \in \mathbb{C}^{m \times n}\).

5. Let \(A \in \mathbb{C}^{m \times n}\) be a matrix with missing entries. More precisely we let \(\Omega \subseteq \{1, \ldots, m\} \times \{1, \ldots, n\}\) be a subset of the row and column indices. We know the value of \(a_{ij}\) if \((i, j) \in \Omega\) but not otherwise. Now one way to recover the matrix \(A\) is to find an \(X \in \mathbb{C}^{m \times n}\) where the loss function \(f\) is minimized, subject to the constraint that \(x_{ij}\) agrees with all known entries of \(A\):

\[
\minimize f(X) \quad \text{subject to} \quad x_{ij} = a_{ij} \text{ for } (i, j) \in \Omega.
\]

One could argue that the most natural candidate for \(f\) is

\[
f(X) = \text{rank}(X), \tag{5.7}
\]

but matrix rank is a discrete valued function and techniques of continuous optimization cannot be applied. A popular alternative is to instead use

\[
f(X) = \|X\|_*
\]

because nuclear norm is the largest convex function that satisfies (3.6). Here we will see how we may nonetheless solve the rank-minimization problem (in principle)

\[
\minimize \text{rank}(X) \quad \text{subject to} \quad x_{ij} = a_{ij} \text{ for } (i, j) \in \Omega, \tag{5.8}
\]

(a) For \(1 \leq r \leq \min(m, n)\), let \(f_r : \mathbb{C}^{m \times n} \to [0, \infty)\) be the function\(^1\)

\[
f_r(X) = \sum_{i=r+1}^{\min(m,n)} \sigma_i(X)^2.
\]

and consider the minimization problem

\[
\minimize f_r(X) \quad \text{subject to} \quad x_{ij} = a_{ij} \text{ for } (i, j) \in \Omega. \tag{5.9}
\]

Let \(X_r\) be a minimizer of (5.9) and \(X_*\) be a minimizer of (5.8). Show that

\[
f_r(X_r) = 0 \quad \text{if and only if} \quad r \geq \text{rank}(X_*).
\]

(b) Deduce that the smallest \(r \in \{1, \ldots, \min(m, n)\}\) such that the minimum value of (5.9) is 0 would have the property that

\[
X_r = X_*.
\]

(c) Implement this strategy in \textsc{Matlab}. Start with \(r = \min(m, n)\) and solve (5.9) using any means you know. If the minimum is 0, reduce \(r\) by 1 and repeat. Keep doing this until you get to a value of \(r\) where the minimum is non-zero. Then the previous value of \(r\) and the corresponding \(X_r\) is the solution to (5.8).

\(^1\)Motivated by the ‘optimization theorist’s numerical rank’ that we discussed in lectures:

\[
\text{rank}_{ot}(A) := \min \left\{ r \in \mathbb{N} \mid \frac{\sum_{i \geq r+1} \sigma_i(A)^2}{\sum_{i \geq 1} \sigma_i(A)^2} \leq \tau \right\}.
\]
(d) Test how well your algorithm works by generating a random matrix $A \in \mathbb{R}^{20 \times 10}$ of rank 5, removing 50% of its entries at random (so $\# \Omega = 100$), and then use your algorithm to find $X_*$. Now check how well $X_*$ agrees with your original $A$ by computing
\[
\frac{\sum_{(i,j) \notin \Omega} (a_{ij} - x_{ij})^2}{\sum_{(i,j) \notin \Omega} a_{ij}^2}.
\] (5.10)
Repeat this experiment 40 times by generating 20 random $A$’s with standard normal entries and another 20 with standard uniform $(0,1)$ entries (i.e. use \texttt{randn} and \texttt{rand} respectively). Record the value of (5.10) and $\text{rank}(X_*)$ each time.

(e) Modify your algorithm so that it now works for $A \in \{1, 2, 3, 4, 5\}^{m \times n}$, i.e. a matrix whose entries are random integers between 1 and 5. Now you need to find some way to round off the entries of your output so that your algorithm yields $X_* \in \{1, 2, 3, 4, 5\}^{m \times n}$. Repeat (d) for 40 random $A \in \{1, 2, 3, 4, 5\}^{20 \times 10}$ (use \texttt{randi} to generate your $A$).