STAT 309: MATHEMATICAL COMPUTATIONS I
FALL 2011
PROBLEM SET 3

MATLAB is recommended for Problems **1** and **2** but you're of course free to use any other programs.

1. Let $A = [a_{ij}]$ be an $n \times n$ matrix with entries

$$a_{ij} = \begin{cases} n + 1 - \max(i, j) & i \le j + 1, \\ 0 & i > j + 1. \end{cases}$$

This is an example of an *upper Hessenberg* matrix: it is upper triangular except that the entries on the subdiagonal $a_{i,i+1}$ may also be non-zero. For $n = 12$ and $n = 25$, do the following:

(a) Compute $\|A\|_\infty$ and $\|A\|_1$.
(b) Compute $\rho(A)$ and $\|A\|_2$.
(c) Using Gerschgorin's theorem, describe the domain that contains all of the eigenvalues.
(d) Compute all of the eigenvalues and singular values of A. How many of the eigenvalues are real and how many are complex?

2. Let $A = [a_{ij}]$ be a 32×32 matrix defined by

$$a_{ij} := \begin{cases} 1 & \text{if } j = i, \\ i - 11 & \text{if } j = i + 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

In other words, A is a bidiagonal matrix with 1's on its diagonal, $-10, -9, \dots, -1, 1, \dots, 21$ on its superdiagonal, and 0's everywhere else.

(a) Construct the Gerschgorin disks for A.
(b) Let $\varepsilon > 0$. Construct a diagonal matrix D so that DAD^{-1} is bidiagonal with 1's on its diagonal, ε's on its superdiagonal, and 0's everywhere else, ie.

$$DAD^{-1} = \begin{bmatrix} 1 & \varepsilon & & & \\ & 1 & \varepsilon & & \\ & & 1 & \ddots & \\ & & & \ddots & \varepsilon \\ & & & & 1 \end{bmatrix}. \tag{2.2}$$

What are the Gerschgorin disks for DAD^{-1}?
(c) Give an algorithm to reduce A in (2.1) to the form in (2.2) with $\varepsilon = 10^{-4}$. How stable is your algorithm?

3. Let $A, B \in \mathbb{R}^{m \times n}$ with $n \le m$. In the lectures, we claim that the solution $Q \in \mathrm{O}(m)$ to

$$\min_{Q^\top Q = I} \|A - BQ\|_F$$

is given by $Q = UV^\top$ where $B^\top A = U\Sigma V^\top$ is its singular value decomposition. Here we will prove it.

Date: October 29, 2011 (Version 1.1); due: November 07, 2011.

(a) Show that
$$\|A - BQ\|_F^2 = \operatorname{tr}(A^\top A) + \operatorname{tr}(B^\top B) - 2\operatorname{tr}(Q^\top B^\top A)$$
and deduce that the minimization problem is equivalent to
$$\max_{Q^\top Q = I} \operatorname{tr}(Q^\top B^\top A).$$

(b) Show that
$$\operatorname{tr}(Q^\top B^\top A) \le \sum_{i=1}^{n} \sigma_i(B^\top A)$$
for any $Q \in \mathrm{O}(n)$. When is the upper bound attained?

(c) Show that
$$\min_{Q^\top Q = I} \|A - BQ\|_F^2 = \sum_{i=1}^{m} (\sigma_i(A)^2 - 2\sigma_i(B^\top A) + \sigma_i(B)^2).$$

(d) Suppose A has full rank. Describe how you would find a symmetric matrix $X \in \mathbb{R}^{n\times n}$ that solves
$$\min_{X = X^\top} \|AX - B\|_F.$$
(*Hint*: Consider the singular value decomposition of A).

4. Let $\mathbf{x} \in \mathbb{C}^m$, $\mathbf{y} \in \mathbb{C}^n$, and $A = \mathbf{x}\mathbf{y}^* \in \mathbb{C}^{m\times n}$.

(a) Show that $\operatorname{rank}(A) = 1$ iff $\mathbf{x}$ and $\mathbf{y}$ are both non-zero. Such a matrix is usually called a rank-1 matrix.

(b) Show that
$$\|A\|_F = \|A\|_2 = \|\mathbf{x}\|_2 \|\mathbf{y}\|_2$$
and that
$$\|A\|_\infty \le \|\mathbf{x}\|_\infty \|\mathbf{y}\|_1.$$
What can you say about $\|A\|_1$?

(c) Let $\mathbf{x}_1, \dots, \mathbf{x}_r \in \mathbb{C}^m$ be linearly independent and $\mathbf{y}_1, \dots, \mathbf{y}_r \in \mathbb{C}^n$ be linearly independent. Let
$$A = \mathbf{x}_1\mathbf{y}_1^* + \cdots + \mathbf{x}_r\mathbf{y}_r^*.$$
Show that $\operatorname{rank}(A) = r$. Show that this is not necessarily true if we drop either of the linear independence conditions.

(d) Given any $0 \ne A \in \mathbb{C}^{m\times n}$, show that
$$\operatorname{rank}(A) = \min\{r \in \mathbb{N} \mid A = \textstyle\sum_{i=1}^{r} \mathbf{x}_i\mathbf{y}_i^*\}.$$
In other words, the rank of a matrix is the smallest r so that it may be expressed as a sum of r rank-1 matrices.

(e) Show that more generally,
$$\|A\|_F \le \sqrt{\operatorname{rank}(A)}\,\|A\|_2.$$
Note that $\rho\operatorname{rank}(A) = \|A\|_F^2 / \|A\|_2^2$ is the 'computer scientist's numerical rank,' one of the three notions of numerical ranks that we discussed.

5. Let $A \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$. We will discuss a variant of $A\mathbf{x} \approx \mathbf{b}$ where the error occurs only in A. Note that in ordinary least squares we assume that the error occurs only in $\mathbf{b}$ while in total least squares we assume that it occurs in both A and $\mathbf{b}$.

(a) Show that if $0 \neq \mathbf{x} \in \mathbb{R}^m$, then

$$\left\| A\left(I - \frac{\mathbf{x}\mathbf{x}^\top}{\mathbf{x}^\top\mathbf{x}}\right)\right\|_F^2 = \|A\|_F^2 - \frac{\|A\mathbf{x}\|_2^2}{\mathbf{x}^\top\mathbf{x}}.$$

(b) Show that the matrix

$$E = \frac{(\mathbf{b} - A\mathbf{x})\mathbf{x}^\top}{\mathbf{x}^\top\mathbf{x}} \in \mathbb{R}^{m\times n}$$

has the smallest 2-norm of all $m \times n$ matrices E that satisfy

$$(A + E)\mathbf{x} = \mathbf{b}.$$

(c) What are the solutions of

$$\min_{(A+E)\mathbf{x}=\mathbf{b}} \|E\|_2 \qquad \text{and} \qquad \min_{(A+E)\mathbf{x}=\mathbf{b}} \|E\|_F?$$

(d) Given $\mathbf{a} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$, and $\delta > 0$. Show how to solve the problems

$$\min_{\|E\|_F\leq\delta} \|E\mathbf{a} - \mathbf{b}\|_2 \qquad \text{and} \qquad \max_{\|E\|_F\leq\delta} \|E\mathbf{a} - \mathbf{b}\|_2$$

over all $E \in \mathbb{R}^{m\times n}$.

6. Let $\mathbf{u} \in \mathbb{C}^n$, $\mathbf{u} \neq \mathbf{0}$. A *Householder* matrix $H_\mathbf{u} \in \mathbb{C}^{n\times n}$ is defined by

$$H_\mathbf{u} = I - \frac{2\mathbf{u}\mathbf{u}^*}{\|\mathbf{u}\|_2^2}.$$

(a) Show that $H_\mathbf{u}$ is both Hermitian and unitary.

(b) Show that for any $\alpha \in \mathbb{C}$, $\alpha \neq 0$,

$$H_{\alpha\mathbf{u}} = H_\mathbf{u}.$$

In other words, $H_\mathbf{u}$ only depends on the 'direction' of $\mathbf{u}$ and not on its 'magnitude'.

(c) In general, given a matrix $M \in \mathbb{C}^{n\times n}$ and a vector $\mathbf{x} \in \mathbb{C}^n$, computing the matrix-vector product $M\mathbf{x}$ requires n inner products — one for each row of M with $\mathbf{x}$. Show that $H_\mathbf{u}\mathbf{x}$ can be computed using only two inner products.

(d) Given $\mathbf{a}, \mathbf{b} \in \mathbb{C}^n$ where $\mathbf{a} \neq e^{i\theta}\mathbf{b}$ for any $\theta \in [0, 2\pi)$ and $\|\mathbf{a}\|_2 = \|\mathbf{b}\|_2$. Find $\mathbf{u} \in \mathbb{C}^n$, $\mathbf{u} \neq \mathbf{0}$ such that

$$H_\mathbf{u}\mathbf{a} = \mathbf{b}.$$

(e) Show that $\mathbf{u}$ is an eigenvector of $H_\mathbf{u}$. What is the corresponding eigenvalue?

(f) Show that every $\mathbf{v} \in \operatorname{span}\{\mathbf{u}\}^\perp$ (cf. orthogonal complement in Homework **2**, Problem **5**) is an eigenvector of $H_\mathbf{u}$. What are the corresponding eigenvalues? What is $\dim(\operatorname{span}\{\mathbf{u}\}^\perp)$?

(g) Find the eigenvalue decomposition of $H_\mathbf{u}$, i.e. find a unitary matrix U and a diagonal matrix Λ such that

$$H_\mathbf{u} = U\Lambda U^*.$$

(*Hint*: Gram-Schmidt algorithm).