STAT 309: MATHEMATICAL COMPUTATIONS I
FALL 2011
PROBLEM SET 2

1. Let $A, B \in \mathbb{C}^{n\times n}$.

(a) Show that if $\mathbf{x}^* A\mathbf{y} = 0$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$, then $A = O$, the zero matrix.
(b) Show that if $\mathbf{x}^* A\mathbf{x} = 0$ for all $\mathbf{x} \in \mathbb{C}^n$, then $A = O$, the zero matrix.
(c) Show that if $\mathbf{x}^* A\mathbf{x} = \mathbf{x}^* B\mathbf{x}$ for all $\mathbf{x} \in \mathbb{C}^n$, then $A = B$.
(d) Show that if $\mathbf{x}^* A\mathbf{x} \in \mathbb{R}$ for all $\mathbf{x} \in \mathbb{C}^n$, then A is Hermitian.
(e) Show that if $\mathbf{x}^* A\mathbf{x} > 0$ for all $\mathbf{0} \neq \mathbf{x} \in \mathbb{C}^n$, then $\|\mathbf{x}\|_A = \sqrt{\mathbf{x}^* A\mathbf{x}}$ defines a norm on $\mathbb{C}^n$.

2. Let $\mathbf{x} \in \mathbb{C}^n$ and $A \in \mathbb{C}^{m\times n}$. We write $\|\mathbf{x}\|_2 = \sqrt{\mathbf{x}^*\mathbf{x}}$ and $\|A\|_2 = \sup_{\|\mathbf{x}\|_2=1} \|A\mathbf{x}\|_2$ for the vector 2-norm and matrix 2-norm respectively.

(a) Show that there is no ambiguity in the notation, i.e. if $A \in \mathbb{C}^{n\times 1} = \mathbb{C}^n$, then $\|A\|_2$ is the same whether we regard it as the vector or matrix 2-norm. What if $A \in \mathbb{C}^{1\times n}$?
(b) Show that the vector 2-norm is unitarily invariant, i.e.
$$\|U\mathbf{x}\|_2 = \|\mathbf{x}\|_2$$
for all unitary matrices $U \in \mathbb{C}^{n\times n}$. Show that no other vector p-norm is unitarily invariant, $1 \leq p \leq \infty$, $p \neq 2$.
(c) Show that the matrix 2-norm is unitarily invariant, i.e.
$$\|UAV\|_2 = \|A\|_2$$
for all unitary matrices $U \in \mathbb{C}^{m\times m}$, $V \in \mathbb{C}^{n\times n}$.
(d) Show that the Frobenius norm is unitarily invariant, i.e.
$$\|UAV\|_F = \|A\|_F$$
for all unitary matrices $U \in \mathbb{C}^{m\times m}$, $V \in \mathbb{C}^{n\times n}$. (*Hint*: First show that $\|A\|_F^2 = \operatorname{tr}(A^*A) = \operatorname{tr}(AA^*)$).
(e) Let $U \in \mathbb{C}^{n\times n}$. Show that the following are equivalent statements:
  (i) $\|U\mathbf{x}\|_2 = \|\mathbf{x}\|_2$ for all $\mathbf{x} \in \mathbb{C}^n$;
  (ii) $(U\mathbf{x})^* U\mathbf{y} = \mathbf{x}^*\mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$;
  (iii) U is unitary.

3. (a) Let $P \in \mathbb{C}^{n\times n}$ be a projection matrix, i.e. $P^2 = P$. Show the eigenvalues of P must be either 0 or 1.
(b) Let $U \in \mathbb{C}^{n\times n}$ be a unitary matrix, i.e. $U^*U = I = UU^*$. Show that the eigenvalues of U must be of the form $e^{i\theta}$ for some $\theta \in [0, 2\pi)$.
(c) Let $H \in \mathbb{C}^{n\times n}$ be a Hermitian matrix, i.e. $H^* = H$. Show that the eigenvalues of H must be real.
(d) Let $S \in \mathbb{C}^{n\times n}$ be a skew-Hermitian matrix, i.e. $S^* = -S$. Show that the eigenvalues of S must be (purely) imaginary.
(e) Let $R \in \mathbb{C}^{n\times n}$ be a triangular matrix. What are the eigenvalues of R? Prove your claim.

4. A square matrix $M \in \mathbb{C}^{n\times n}$ is called *positive semidefinite* if
$$\mathbf{x}^* M\mathbf{x} \geq 0$$

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for all $\mathbf{x} \in \mathbb{C}^n$. $M \in \mathbb{C}^{n\times n}$ is called *positive definite* if (i) M is positive semidefinite; and (ii) $\mathbf{x}^* M \mathbf{x} = 0$ only if $\mathbf{x} = \mathbf{0}$.

(a) Show that every positive definite matrix is nonsingular (i.e. invertible).
(b) Show that if M is positive semidefinite and $\lambda \in \mathbb{C}$ is an eigenvalue of M, then $\lambda \in \mathbb{R}$ and $\lambda \geq 0$. If it is positive definite, then $\lambda > 0$.
(c) Is it possible for a non-Hermitian matrix M to be positive definite?
(d) Let $A \in \mathbb{C}^{m\times n}$. Show that A^*A and AA^* are positive semidefinite matrices. Hence deduce that singular values are always nonnegative.
(e) Let $A \in \mathbb{C}^{m\times n}$ be full-rank. Show that either A^*A or AA^* is a positive definite matrix.

5. Let W be a subspace of $\mathbb{C}^n$. The subspace $W^\perp$ is called the *orthogonal complement* of W.

$$W^\perp = \{\mathbf{v} \in \mathbb{C}^n \mid \mathbf{v}^*\mathbf{w} = 0 \text{ for all } \mathbf{w} \in W\}.$$

(a) Show that $\mathbb{C}^n = W \oplus W^\perp$ and that $W = (W^\perp)^\perp$.
(b) Let $A \in \mathbb{C}^{m\times n}$. Show that

$$\ker(A^*) = \operatorname{im}(A)^\perp \quad \text{and} \quad \operatorname{im}(A^*) = \ker(A)^\perp.$$

(c) Deduce the Fredholm alternative:

$$\mathbb{C}^m = \ker(A^*) \oplus \operatorname{im}(A) \quad \text{and} \quad \mathbb{C}^n = \operatorname{im}(A^*) \oplus \ker(A).$$

(d) Let $A \in \mathbb{C}^{n\times n}$ be normal, i.e. $A^*A = AA^*$. Show that

$$\ker(A^*) = \ker(A) \quad \text{and} \quad \operatorname{im}(A^*) = \operatorname{im}(A).$$

(e) Deduce that for a normal matrix,

$$\mathbb{C}^m = \ker(A) \oplus \operatorname{im}(A).$$

6. Recall that for $A \in \mathbb{C}^{m\times n}$ with $\operatorname{rank}(A) = r$, we call $A = U\Sigma V^*$ with $U \in \mathbb{C}^{m\times m}$, $\Sigma \in \mathbb{C}^{m\times n}$, $V \in \mathbb{C}^{n\times n}$ the *full* singular value decomposition and $A = U\Sigma V^*$ with $U \in \mathbb{C}^{m\times r}$, $\Sigma \in \mathbb{C}^{r\times r}$, $V \in \mathbb{C}^{n\times r}$ the *condensed* singular value decomposition (where U, Σ, V have the requisite properties).

(a) Find a full singular value decomposition of the matrix $\mathbf{x} \in \mathbb{C}^m = \mathbb{C}^{m\times 1}$. Is this decomposition unique?
(b) Let $\mathbf{x} \in \mathbb{C}^m$ and $\mathbf{y} \in \mathbb{C}^n$. Find a full singular value decomposition of the matrix $\mathbf{x}\mathbf{y}^* \in \mathbb{C}^{m\times n}$. Is this decomposition unique?
(c) Let $A \in \mathbb{C}^{n\times n}$ be Hermitian. How is the singular value decomposition of A related to its eigenvalue decomposition?
(d) Show that if all singular values of A are distinct and $m = n$, then its singular value decomposition is unique.
(e) Show that if all non-zero singular values of A are distinct, then its condensed singular value decomposition is unique.