STAT 309: MATHEMATICAL COMPUTATIONS I
FALL 2011
PROBLEM SET 1

1. Here is another way to derive the normal equation without using any calculus. Recall that the null space or kernel of a matrix $A \in \mathbb{R}^{m\times n}$ is the set

$$\ker(A) = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}$$

while the range space or image is the set

$$\operatorname{im}(A) = \{\mathbf{y} \in \mathbb{R}^m \mid \mathbf{y} = A\mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{R}^n\}$$

and $\mathbf{b} \in \mathbb{R}^m$.

(a) Show that

$$\ker(A^\top A) = \ker(A).$$

(b) Show that

$$\operatorname{im}(A^\top A) = \operatorname{im}(A^\top).$$

(c) Deduce that

$$A^\top A\mathbf{x} = A^\top \mathbf{b}$$

always has a solution. We call this the normal equation.

(d) Give an example where $A\mathbf{x} = \mathbf{b}$ has no solution but $A^\top A\mathbf{x} = A^\top \mathbf{b}$ has a solution.

(e) Show that (a), (b), and (c) are false in general over a field with two elements $\mathbb{F}_2 = \{0, 1\}$ with arithmetic done modulo 2.

2. We would like to solve the differential equation

$$\begin{cases} -v''(x) = \dfrac{m\omega^2}{k} v(x), & 0 < x < 1, \\ v(0) = 0, \ v(1) = 0. \end{cases}$$

This comes up when studying a vibrating string with m the *mass* per unit length and k the *stiffness* per unit length, both positive constants. We need to determine the function $v : [0, 1] \to \mathbb{R}$ and the number $\omega \in \mathbb{R}$. Here $v(x)$ gives us the *amplitude* of the string at x and and ω gives us the *vibration frequency* of the string.

(a) Following the technique used in Lecture 3, show that we may discretize the differential equation into the following difference equation

$$\begin{cases} \dfrac{-v_{i-1} + 2v_i - v_{i+1}}{n^{-2}} = \lambda v_i, & 1 \le i \le n-1, \\ v_0 = 0, \ v_n = 0. \end{cases}$$

(b) Show that the difference equation can be rewritten as an eigenvalue problem

$$A\mathbf{v} = \lambda\mathbf{v}$$

where λ is an approximation of $m\omega^2/k$.

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3. Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}$. Suppose we would like to *learn* a function $f : X \to Y$ from a *training set* of data $\{(\mathbf{x}_i, y_i) \in X \times Y \mid i = 1, \dots, m\}$. We will assume that f can be expressed as a linear combination

$$f(\mathbf{x}) = \sum_{i=1}^{m} c_i K(\mathbf{x}, \mathbf{x}_i)$$

where $K(\mathbf{x}, \mathbf{y}) = \exp(-\|\mathbf{x} - \mathbf{y}\|_2^2 / 2\sigma^2)$ is a *Gaussian kernel*. Following the data fitting technique discussed in Lecture 3, describe how one may determine the coefficients $c_1, \dots, c_m \in \mathbb{R}$ by solving a least squares problem. You will need to describe the least squares problem explicitly: What are the coefficient matrix and the right-hand side.

4. In testing your codes, it is often important to know how to randomly generate matrices with some specified properties. In MATLAB, you can generate a random $m \times n$ matrix X with built-in functions `rand(m,n)` and `randn(m,n)`, where the entries are drawn respectively from the uniform distribution on the interval $(0, 1)$ and the standard normal distribution. For each of the following, write a program that will generate:

(a) $n \times n$ real symmetric matrices, i.e. $X^\top = X$;
(b) $n \times n$ real skew-symmetric matrix, i.e. $X^\top = -X$;
(c) $n \times n$ non-singular matrices (a.k.a. invertible matrices);
(d) $n \times n$ symmetric positive definite Toeplitz matrices;
(e) $m \times n$ matrices of rank r, where $r \in \{0, 1, \dots, \min(m, n)\}$ is an unspecified input;
(f) $m \times n$ matrices whose entries are uniformly distibuted in $[\alpha, \beta]$, where $\alpha < \beta$ are unspecified inputs;
(g) $m \times n$ matrices whose entries are normally distributed with mean μ and variance σ^2, where μ and σ are unspecified inputs;
(h) $m \times n$ matrices whose entries are either 0 or 1 with probabilities p and $1 - p$ respectively, where $p \in (0, 1)$ is an unspecified input.