This homework mostly serves as a linear algebra refresher. We will recall some definitions. The null space or kernel of a matrix $A \in \mathbb{R}^{m \times n}$ is the set

$$\ker(A) = \{ x \in \mathbb{R}^n : Ax = 0 \}$$

while the range space or image is the set

$$\text{im}(A) = \{ y \in \mathbb{R}^m : y = Ax \text{ for some } x \in \mathbb{R}^n \}.$$ 

The rank and nullity of $A$ are defined as the dimensions of these spaces,

$$\text{rank}(A) = \dim \text{im}(A) \quad \text{and} \quad \text{nullity}(A) = \dim \ker(A).$$

By convention we write all vectors in $\mathbb{R}^n$ as column vectors.

1. (a) For $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$, show that

$$\text{im}(AB) \subseteq \text{im}(A) \quad \text{and} \quad \ker(AB) \supseteq \ker(B).$$

When does equality occur in each of these inclusions?

(b) For $A, B \in \mathbb{R}^{n \times n}$, show that

$$\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\},$$

$$\text{nullity}(AB) \leq \text{nullity}(A) + \text{nullity}(B),$$

$$\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B).$$

(c) For $A, B \in \mathbb{R}^{n \times n}$, show that if $AB = 0$, then

$$\text{rank}(A) + \text{rank}(B) \leq n.$$

2. (a) Let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{p \times q}$. Show that

$$\text{rank} \left( \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right) = \text{rank}(A) + \text{rank}(B).$$

We have used the block matrix notation here. For example if $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \in \mathbb{R}^{2 \times 3}$ and $B = \begin{bmatrix} \beta \end{bmatrix} \in \mathbb{R}^{2 \times 1}$, then

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} a & b & c & 0 \\ d & e & f & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & \beta \end{bmatrix} \in \mathbb{R}^{4 \times 4}.$$

This is sometimes also denoted as $A \oplus B$. It is a direct sum of operators induced by a direct sum of vector spaces.

(b) For $x = [x_1, \ldots, x_m]^T \in \mathbb{R}^m$ and $y = [y_1, \ldots, y_n]^T \in \mathbb{R}^n$, observe that $xy^T \in \mathbb{R}^{m \times n}$. Let $A \in \mathbb{R}^{m \times n}$. Show that $\text{rank}(A) = 1$ iff $A = xy^T$ for some nonzero $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$. 

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3. Let $A \in \mathbb{R}^{m \times n}$.
   (a) Show that
   $$\ker(A^T A) = \ker(A) \quad \text{and} \quad \im(A^T A) = \im(A^T).$$
   Give an example to show this is not true over a finite field (e.g. a field of two elements $\mathbb{F}_2 = \{0, 1\}$ with binary arithmetic).
   (b) Show that
   $$A^T A x = A^T b$$
   always has a solution (even if $A x = b$ has no solution). Give an example to show that this is not true over a finite field.

4. Let $v_1, \ldots, v_n \in \mathbb{R}^n$. Let $G_r = [g_{ij}] \in \mathbb{R}^{r \times r}$ be the matrix with
   $$g_{ij} = v_i^T v_j$$
   for $i, j = 1, \ldots, r$. This is called a Gram matrix.
   (a) Show that $v_1, \ldots, v_r$ are linearly independent iff $\text{nullity}(G_r) = 0$.
   (b) Show that $G_r = I_r$ iff $v_1, \ldots, v_r$ are pairwise orthogonal unit vectors, i.e., $\|v_i\|_2 = 1$ for all $i = 1, \ldots, r$, and $v_i^T v_j = 0$ for all $i \neq j$. If this holds, show that
   $$\sum_{i=1}^r (v^T v_i)^2 \leq \|v\|_2^2 \tag{4.1}$$
   for all $v \in \mathbb{R}^n$. What can you say about $v_1, \ldots, v_r$ if equality always holds in (4.1) for all $v \in \mathbb{R}^n$?

5. Let $A \in \mathbb{C}^{n \times n}$. Recall that $A$ is diagonalizable iff there exists an invertible $X \in \mathbb{C}^{n \times n}$ such that $X^{-1} A X = \Lambda$, a diagonal matrix.
   (a) Show that $A$ is diagonalizable if and only if its minimal polynomial is of the form
   $$m_A(x) = (x - \lambda_1) \cdots (x - \lambda_d)$$
   where $\lambda_1, \ldots, \lambda_d \in \mathbb{C}$ are all distinct. Hence deduce for a diagonalizable matrix, the degree of its minimal polynomial equals the number of distinct eigenvalues.
   (b) Let $A$ be diagonalizable. Let $x_1, \ldots, x_n \in \mathbb{C}^n$ be $n$ linearly independent right eigenvectors, i.e., $A x_i = \lambda_i x_i$; and $y_1, \ldots, y_n \in \mathbb{C}^n$ be $n$ linearly independent left eigenvectors, i.e., $y_i^T A = \lambda_i y_i^T$. Show that there is a choice of left and right eigenvectors of $A$ such that any vector $v \in \mathbb{C}^n$ can be expressed as
   $$v = \sum_{i=1}^n (y_i^T v) x_i.$$
   If we write $X = [x_1, \ldots, x_n] \in \mathbb{C}^{n \times n}$ and $Y = [y_1, \ldots, y_n] \in \mathbb{C}^{n \times n}$. What is the relation between $X$ and $Y$?