STAT 280: OPTIMIZATION
WINTER 2021
PROBLEM SET 2

1. (a) Show that for any $X \in \mathbb{R}^{n\times n}$, the series

$$I + X + \frac{X^2}{2} + \cdots + \frac{X^k}{k!} + \cdots$$

is always convergent. The limit of this series is usually denoted as $\exp(X)$ and called matrix exponential of X.

(b) Show that the set of invertible matrices

$$\mathrm{GL}(n) \coloneqq \{X \in \mathbb{R}^{n\times n} : \det(X) \neq 0\}$$

is an open set in $\mathbb{R}^{n\times n}$.

(c) Show that the set of nilpotent matrices $\{X \in \mathbb{R}^{n\times n} : X^k = 0 \text{ for some } k \in \mathbb{N} \cup \{0\}\}$ is a closed set in $\mathbb{R}^{n\times n}$.

(d) Show that the set of orthogonal matrices

$$\mathrm{O}(n) \coloneqq \{X \in \mathbb{R}^{n\times n} : X^{\mathsf{T}}X = I\}$$

is a compact set in $\mathbb{R}^{n\times n}$.

(e) Show that the set of symmetric positive definite matrices

$$\mathbb{S}^n_{++} \coloneqq \{X \in \mathbb{S}^n : \mathbf{y}^{\mathsf{T}}X\mathbf{y} > 0 \text{ for all } \mathbf{0} \neq \mathbf{y} \in \mathbb{R}^n\}$$

is an open set in $\mathbb{S}^n \coloneqq \{X \in \mathbb{R}^{n\times n} : X^{\mathsf{T}} = X\}$ and that its closure is the set of symmetric positive semidefinite matrices

$$\mathbb{S}^n_{+} \coloneqq \{X \in \mathbb{S}^n : \mathbf{y}^{\mathsf{T}}X\mathbf{y} \geq 0 \text{ for all } \mathbf{y} \in \mathbb{R}^n\}.$$

2. (a) Let $f : \mathbb{R}^n \to \mathbb{R}$ be in $C^1(\mathbb{R}^n)$. Suppose f satisfies

$$\left|\frac{\partial f}{\partial x_i}(\mathbf{x})\right| \leq K, \quad i = 1, \ldots, n,$$

for all $\mathbf{x} \in \mathbb{R}^n$ where $K > 0$ is a constant. Show that

$$|f(\mathbf{x}) - f(\mathbf{y})| \leq \sqrt{n}K\|\mathbf{x} - \mathbf{y}\|$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ (Hint: Apply the univariate mean-value theorem n times).

(b) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} \left(1 - \cos\dfrac{x^2}{y}\right)\sqrt{x^2 + y^2} & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$$

Show that f is continuous but not differentiable at $(0, 0)$. For which $\mathbf{v} \in \mathbb{R}^2$ does the directional derivative $\partial_{\mathbf{v}} f(0, 0)$ exist?

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3. Let $A : \mathbb{R} \to \mathbb{R}^{n \times n}$ be the matrix-valued function

$$A(x) = \begin{bmatrix} a_{11}(x) & \cdots & a_{1n}(x) \\ \vdots & & \vdots \\ a_{n1}(x) & \cdots & a_{nn}(x) \end{bmatrix}$$

where the functions $a_{ij} : \mathbb{R} \to \mathbb{R}$ are all in $C^1(\mathbb{R})$ for all $i, j = 1, \ldots, n$.

(a) Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \operatorname{tr}\big(A(x)^3\big)$ is differentiable and that

$$f'(x) = 3 \operatorname{tr}\big(A(x)^2 A'(x)\big)$$

where

$$A'(x) = \begin{bmatrix} a'_{11}(x) & \cdots & a'_{1n}(x) \\ \vdots & & \vdots \\ a'_{n1}(x) & \cdots & a'_{nn}(x) \end{bmatrix}.$$

(b) Let $n = 2$. Suppose $A(x) \in \mathbb{S}^2_{++}$ for all $x \in \mathbb{R}$ and define $B : \mathbb{R} \to \mathbb{S}^{2\times 2}$ by $B(x) = A(x)^{-1}$. Show that

$$\frac{d}{dx} \log \det A(x) = \sum\nolimits_{i,j=1}^{2} a'_{ij}(x) b_{ij}(x).$$

This is actually true for arbitrary n. For $n = 1$, it reduces to $(\log a(x))' = a'(x)/a(x)$.

4. Find the gradients of the following functions without calculating a single partial derivative.

(a) $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(\mathbf{x}) = \|\mathbf{x}\|.$$

Find the set of points where f is not differentiable.

(b) $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(\mathbf{x}) = \frac{\mathbf{x}^\mathsf{T} A \mathbf{x}}{\mathbf{x}^\mathsf{T} B \mathbf{x}}$$

where $A, B \in \mathbb{S}^n_{++}$.

(c) $f : \mathbb{S}^n_{++} \to \mathbb{R}$ defined by

$$f(X) = \det(X).$$

(d) $f : \mathbb{S}^n_{++} \to \mathbb{R}$ defined by

$$f(X) = \frac{\det(X)}{\operatorname{tr}(X)}.$$

(e) $f : \mathbb{R}^n \times \mathbb{S}^n_{++} \to \mathbb{R}$ defined by

$$f(\mathbf{x}, Y) = \mathbf{x}^\mathsf{T} Y^{-1} \mathbf{x}.$$

(f) $f : \Omega \to \mathbb{R}$ defined by

$$\Omega = \{X \in \mathbb{R}^{m \times n} : X^\mathsf{T} A X + B^\mathsf{T} X + X^\mathsf{T} B + C \succ 0\}$$

with $A \in \mathbb{S}^m$, $B \in \mathbb{R}^{m \times n}$, $C \in \mathbb{S}^n$ arbitrary; and

$$f(X) = \log \det(X^\mathsf{T} A X + B^\mathsf{T} X + X^\mathsf{T} B + C).$$

5. Find the derivatives of the following functions without calculating a single partial derivative.

(a) $f : \mathbb{R}^4 \to \mathbb{R}^{2\times 2}$ defined by

$$f(x, y, z, w) = \begin{bmatrix} x & y \\ z & w \end{bmatrix}^2 + \begin{bmatrix} x & y \\ z & w \end{bmatrix}^\mathsf{T} = X^2 + X^\mathsf{T}.$$

(b) $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ defined by

$$f(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|}.$$

(c) $f : \mathbb{R}^{m\times n} \setminus \{0\} \to \mathbb{R}^{m\times n} \setminus \{0\}$ defined by

$$f(X) = \frac{X}{\|X\|}.$$

(d) $f : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ defined by

$$f(X) = X^4.$$

(e) $f : \mathrm{GL}(n) \to \mathrm{GL}(n)$ defined by

$$f(X) = X^{-1}.$$

(f) $f : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ defined by

$$f(X) = \exp(X),$$

as in Problem **1**(a).