1. (a) Show that for any $X \in \mathbb{R}^{n \times n}$, the series
\[ I + X + \frac{X^2}{2} + \cdots + \frac{X^k}{k!} + \cdots \]
is always convergent. The limit of this series is usually denoted as $\exp(X)$ and called matrix exponential of $X$.

(b) Show that the set of invertible matrices
\[ \text{GL}(n) := \{ X \in \mathbb{R}^{n \times n} : \det(X) \neq 0 \} \]
is an open set in $\mathbb{R}^{n \times n}$.

(c) Show that the set of nilpotent matrices $\{ X \in \mathbb{R}^{n \times n} : X^k = 0 \text{ for some } k \in \mathbb{N} \cup \{0\} \}$ is a closed set in $\mathbb{R}^{n \times n}$.

(d) Show that the set of orthogonal matrices
\[ \text{O}(n) := \{ X \in \mathbb{R}^{n \times n} : X^T X = I \} \]
is a compact set in $\mathbb{R}^{n \times n}$.

(e) Show that the set of symmetric positive definite matrices
\[ \mathbb{S}^n_{++} := \{ X \in \mathbb{S}^n : y^T X y > 0 \text{ for all } 0 \neq y \in \mathbb{R}^n \} \]
is an open set in $\mathbb{S}^n := \{ X \in \mathbb{R}^{n \times n} : X^T = X \}$ and that its closure is the set of symmetric positive semidefinite matrices
\[ \mathbb{S}^n_+ := \{ X \in \mathbb{S}^n : y^T X y \geq 0 \text{ for all } y \in \mathbb{R}^n \} \].

2. (a) Let $f : \mathbb{R}^n \to \mathbb{R}$ be in $C^1(\mathbb{R}^n)$. Suppose $f$ satisfies
\[ \left| \frac{\partial f}{\partial x_i}(x) \right| \leq K, \quad i = 1, \ldots, n, \]
for all $x \in \mathbb{R}^n$ where $K > 0$ is a constant. Show that
\[ |f(x) - f(y)| \leq \sqrt{n} K \|x - y\| \]
for all $x, y \in \mathbb{R}^n$ (Hint: Apply the univariate mean-value theorem $n$ times).

(b) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by
\[ f(x, y) = \begin{cases} 
1 - \cos \frac{x^2}{y} \sqrt{x^2 + y^2} & \text{if } y \neq 0, \\
0 & \text{if } y = 0.
\end{cases} \]
Show that $f$ is continuous but not differentiable at $(0,0)$. For which $v \in \mathbb{R}^2$ does the directional derivative $\partial_v f(0,0)$ exist?
3. Let \( A : \mathbb{R} \to \mathbb{R}^{n \times n} \) be the matrix-valued function

\[
A(x) = \begin{bmatrix}
a_{11}(x) & \cdots & a_{1n}(x) \\
\vdots & \ddots & \vdots \\
a_{n1}(x) & \cdots & a_{nn}(x)
\end{bmatrix}
\]

where the functions \( a_{ij} : \mathbb{R} \to \mathbb{R} \) are all in \( C^1(\mathbb{R}) \) for all \( i, j = 1, \ldots, n \).

(a) Show that the function \( f : \mathbb{R} \to \mathbb{R} \) defined by 

\[
f(x) = \text{tr}(A(x)^2)
\]

is differentiable and that 

\[
f'(x) = 3 \text{tr}(A(x)^2 A'(x))
\]

where

\[
A'(x) = \begin{bmatrix}
a'_{11}(x) & \cdots & a'_{1n}(x) \\
\vdots & \ddots & \vdots \\
a'_{n1}(x) & \cdots & a'_{nn}(x)
\end{bmatrix}.
\]

(b) Let \( n = 2 \). Suppose \( A(x) \in \mathbb{S}^2_{++} \) for all \( x \in \mathbb{R} \) and define \( B : \mathbb{R} \to \mathbb{S}^{2 \times 2} \) by \( B(x) = A(x)^{-1} \).

Show that 

\[
\frac{d}{dx} \log \det A(x) = \sum_{i,j=1}^2 a'_{ij}(x) b_{ij}(x).
\]

This is actually true for arbitrary \( n \). For \( n = 1 \), it reduces to \((\log a(x))' = a'(x)/a(x)\).

4. Find the gradients of the following functions without calculating a single partial derivative.

(a) \( f : \mathbb{R}^n \to \mathbb{R} \) defined by 

\[
f(x) = \|x\|.
\]

Find the set of points where \( f \) is not differentiable.

(b) \( f : \mathbb{R}^n \to \mathbb{R} \) defined by 

\[
f(x) = \frac{x^T A x}{x^T B x}
\]

where \( A, B \in \mathbb{S}^n_{++} \).

(c) \( f : \mathbb{S}^n_{++} \to \mathbb{R} \) defined by 

\[
f(X) = \det(X).
\]

(d) \( f : \mathbb{S}^n_{++} \to \mathbb{R} \) defined by 

\[
f(X) = \frac{\det(X)}{\text{tr}(X)}.
\]

(e) \( f : \mathbb{R}^n \times \mathbb{S}^n_{++} \to \mathbb{R} \) defined by 

\[
f(x, Y) = x^T Y^{-1} x.
\]

(f) \( f : \Omega \to \mathbb{R} \) defined by 

\[
\Omega = \{X \in \mathbb{R}^{m \times n} : X^T A X + B^T X + X^T B + C > 0 \}
\]

with \( A \in \mathbb{S}^m \), \( B \in \mathbb{R}^{m \times n} \), \( C \in \mathbb{S}^n \) arbitrary; and 

\[
f(X) = \log \det(X^T A X + B^T X + X^T B + C).
\]

5. Find the derivatives of the following functions without calculating a single partial derivative.

(a) \( f : \mathbb{R}^4 \to \mathbb{R}^{2 \times 2} \) defined by 

\[
f(x, y, z, w) = \begin{bmatrix} x & y \\ z & w \end{bmatrix}^2 + \begin{bmatrix} x & y \\ z & w \end{bmatrix}^T = X^2 + X^T.
\]

(b) \( f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\} \) defined by 

\[
f(x) = \frac{x}{\|x\|}.
\]
(c) $f : \mathbb{R}^{m \times n} \setminus \{0\} \to \mathbb{R}^{m \times n} \setminus \{0\}$ defined by
$$f(X) = \frac{X}{\|X\|}.$$

(d) $f : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$ defined by
$$f(X) = X^4.$$

(e) $f : \text{GL}(n) \to \text{GL}(n)$ defined by
$$f(X) = X^{-1}.$$

(f) $f : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$ defined by
$$f(X) = \exp(X),$$
    as in Problem 1(a).