CHAPTER 14

Implicit Function Theorems and Lagrange Multipliers

14.1. The Implicit Function Theorem for a Single Equation

Suppose we are given a relation in $\mathbb{R}^2$ of the form

$$F(x, y) = 0. \tag{14.1}$$

Then to each value of x there may correspond one or more values of y which satisfy (14.1)—or there may be no values of y which do so. If $I = \{x: x_0 - h < x < x_0 + h\}$ is an interval such that for each $x \in I$ there is exactly one value of y satisfying (14.1), then we say that $F(x, y) = 0$ defines y as a function of x **implicitly** on I. Denoting this function by f, we have $F[x, f(x)] = 0$ for x on I.

An Implicit function theorem is one which determines conditions under which a relation such as (14.1) defines y as a function of x or x as a function of y. The solution is a local one in the sense that the size of the interval I may be much smaller than the domain of the relation F. Figure 14.1 shows the graph of a relation such as (14.1). We see that F defines y as a function of x in a region about P, but not beyond the point Q. Furthermore, the relation does not yield y as a function of x in any region containing the point Q in its interior.

The simplest example of an Implicit function theorem states that if F is smooth and if P is a point at which $F_{,2}$ (that is, $\partial F/\partial y$) does not vanish, then it is possible to express y as a function of x in a region containing this point. More precisely we have the following result.

Theorem 14.1. *Suppose that F, $F_{,1}$ and $F_{,2}$ are continuous on an open set A in $\mathbb{R}^2$ containing the point $P(x_0, y_0)$, and suppose that*

$$F(x_0, y_0) = 0, \qquad F_{,2}(x_0, y_0) \neq 0.$$

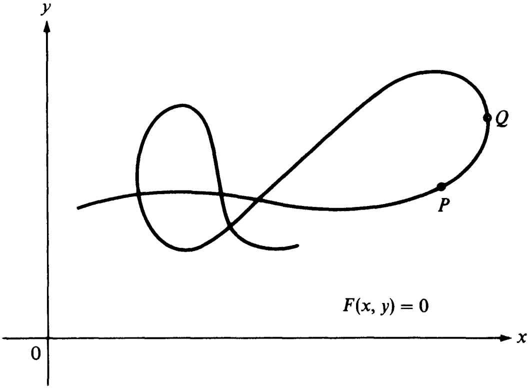

Figure 14.1

(a) *Then there are positive numbers h and k which determine a rectangle R contained in A* (*see Figure* 14.2) *given by*

$$R = \{(x, y): |x - x_0| < h, |y - y_0| < k\},$$

such that for each x in $I = \{x: |x - x_0| < h\}$ *there is a unique number y in* $J = \{y: |y - y_0| < k\}$ *which satisfies the equation* $F(x, y) = 0$. *The totality of the points* (x, y) *forms a function f whose domain contains I and whose range is in J.*

(b) *The function f and its derivative f′ are continuous on I.*

We shall give two proofs of Part (a), one which uses the elementary proper-

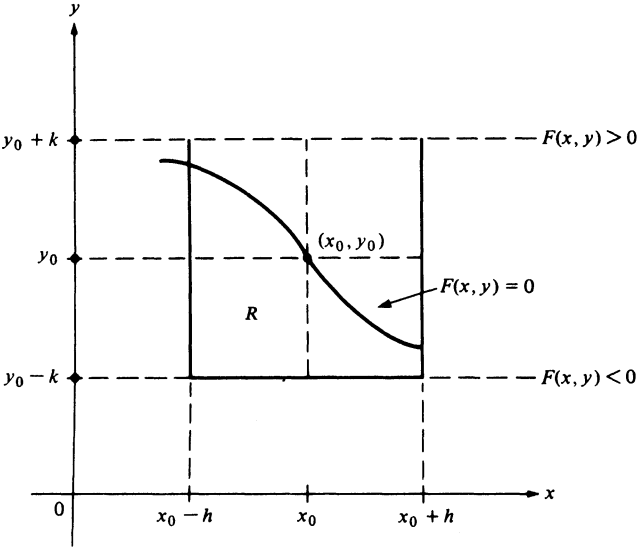

Figure 14.2

ties of continuous functions and the Intermediate-value theorem (Theorem 3.3), and a second which employs the Fixed point theorem in Chapter 13 (Theorem 13.2).

First Proof of Part (a). We assume $F_{,2}(x_0, y_0) > 0$; otherwise we replace F by $-F$ and repeat the argument. Since $F_{,2}$ is continuous there is a (sufficiently small) square $S = \{(x, y): |x - x_0| \leqslant k, |y - y_0| \leqslant k\}$ which is contained in A and on which $F_{,2}$ is positive. For each fixed value of x such that $|x - x_0| < k$ we see that $F(x, y)$, considered as a function of y, is an increasing function. Since $F(x_0, y_0) = 0$, it is clear that

$$F(x_0, y_0 + k) > 0 \quad \text{and} \quad F(x_0, y_0 - k) < 0.$$

Because F is continuous on S, there is a (sufficiently small) number h such that $F(x, y_0 + k) > 0$ on $I = \{x: |x - x_0| < h\}$ and $F(x, y_0 - k) < 0$ on I. We fix a value of x in I and examine solutions of $F(x, y) = 0$ in the rectangle R (see Figure 14.2). Since $F(x, y_0 - k)$ is negative and $F(x, y_0 + k)$ is positive, there is a value $\bar{y}$ in R such that $F(x, \bar{y}) = 0$. Also, because $F_{,2} > 0$, there is precisely one such value. The correspondence $x \to \bar{y}$ is the function we seek, and we denote it by f. □

(b) To show that f is continuous at x_0 let $\varepsilon > 0$ be given and suppose that ε is smaller than k. Then we may construct a square S_ε with side 2ε and center at (x_0, y_0) as in the proof of Part (a). There is a value $h' < h$ such that f is a function on $I' = \{x: |x - x_0| < h'\}$. Therefore

$$|f(x) - f(x_0)| < \varepsilon \quad \text{whenever} \quad |x - x_0| < h',$$

and f is continuous at x_0. At any other point $x_1 \in I$, we construct a square S_1 with center at $(x_1, f(x_1))$ and repeat the above argument.

To show that f' exists and is continuous we use the Fundamental lemma on differentiation (Theorem 7.2). Let $x \in I$ and choose a number ρ such that $x + \rho \in I$. Then

$$F(x + \rho, f(x + \rho)) = 0 \quad \text{and} \quad F(x, f(x)) = 0.$$

Writing $f(x + \rho) = f + \Delta f$ and using Theorem 7.2, we obtain

$$[F_{,1}(x, f) + \varepsilon_1(\rho, \Delta f)]\rho + [F_{,2}(x, f) + \varepsilon_2(\rho, \Delta f)]\Delta f = 0 \tag{14.2}$$

where ε_1 and ε_2 tend to zero as $\rho, \Delta f \to 0$. From the continuity of f, which we established, it follows that $\Delta f \to 0$ as $\rho \to 0$. From (14.2) it is clear that

$$\frac{\Delta f}{\rho} = \frac{f(x + \rho) - f(x)}{\rho} = -\frac{F_{,1}(x, f) + \varepsilon_1(\rho, \Delta f)}{F_{,2}(x, f) + \varepsilon_2(\rho, \Delta f)}.$$

Since the right side tends to a limit as $\rho \to 0$, we see that

$$f'(x) = -\frac{F_{,1}(x, f)}{F_{,2}(x, f)}. \tag{14.3}$$

By hypothesis the right side of (14.3) is continuous, and so f' is also. □

SECOND PROOF OF PART (a). For fixed x in the rectangle R we consider the mapping

$$T_x y = y - \frac{F(x, y)}{F_{,2}(x_0, y_0)},$$

which takes a point y in J into $\mathbb{R}^1$. We shall show that for h and k sufficiently small, the mapping takes J into J and has a fixed point. That is, there is a y such that $T_x y = y$ or, in other words, there is a y such that $F(x, y) = 0$. To accomplish this, we first write the mapping T_x in the more complicated form:

$$T_x y = y_0 - \frac{F_{,1}(x_0, y_0)}{F_{,2}(x_0, y_0)}(x - x_0)$$
$$- \frac{1}{F_{,2}(x_0, y_0)}[F(x, y) - F_{,1}(x_0, y_0)(x - x_0) - F_{,2}(x_0, y_0)(y - y_0)].$$

We define

$$c = \frac{F_{,1}(x_0, y_0)}{F_{,2}(x_0, y_0)}, \qquad \psi(x, y) = \frac{1}{F_{,2}(x_0, y_0)}[F(x, y) - F_{,1}(x_0, y_0)(x - x_0)$$
$$- F_{,2}(x_0, y_0)(y - y_0)].$$

Then the mapping $T_x y$ can be written

$$T_x y = y_0 - c(x - x_0) - \psi(x, y).$$

Since $F(x_0, y_0) = 0$, we see that

$$\psi(x_0, y_0) = 0, \qquad \psi_{,1}(x_0, y_0) = 0, \qquad \psi_{,2}(x_0, y_0) = 0.$$

Because $\psi_{,1}$ and $\psi_{,2}$ are continuous we can take k so small that

$$|\psi_{,1}(x, y)| \leqslant \tfrac{1}{2}, \qquad |\psi_{,2}(x, y)| \leqslant \tfrac{1}{2},$$

for (x, y) in the square $S = \{(x, y): |x - x_0| \leqslant k, |y - y_0| \leqslant k\}$. We now expand $\psi(x, y)$ in a Taylor series in S about the point (x_0, y_0) getting

$$\psi(x, y) = \psi_{,1}(\xi, \eta)(x - x_0) + \psi_{,2}(\xi, \eta)(y - y_0), \qquad (\xi, \eta) \in S.$$

Hence for $h \leqslant k$, we have the estimate in the rectangle R:

$$|\psi(x, y)| \leqslant \tfrac{1}{2}h + \tfrac{1}{2}k.$$

Next we show that if we reduce h sufficiently, the mapping T_x takes the interval (space) J into J. We have

$$|T_x y - y_0| \leqslant |c(x - x_0)| + |\psi(x, y)|$$
$$\leqslant |c|h + \tfrac{1}{2}h + \tfrac{1}{2}k = (\tfrac{1}{2} + |c|)h + \tfrac{1}{2}k.$$

We choose h so small that $(\frac{1}{2} + |c|)h \leqslant k$. Then $T_x y$ maps J into J for each x in $I = \{x: |x - x_0| \leqslant h\}$. The mapping T_x is a contraction map; in fact, by the Mean-value theorem

$$|T_x y_1 - T_x y_2| = |-\psi(x, y_1) + \psi(x, y_2)| \leqslant \tfrac{1}{2}|y_1 - y_2|.$$

We apply Theorem 13.2 and for each fixed x in I, there is a unique y in J such that $F(x, y) = 0$. That is, y is a function of x for $(x, y) \in R$. □

The Implicit function theorem has a number of generalizations and applications. If F is a function from $\mathbb{R}^{N+1}$ to $\mathbb{R}^1$, we may consider whether or not the relation $F(x_1, x_2, \ldots, x_N, y) = 0$ defines y as a function from $\mathbb{R}^N$ into $\mathbb{R}^1$. That is, we state conditions which show that $y = f(x_1, x_2, \ldots, x_N)$. The proof of the following theorem is a straightforward extension of the proof of Theorem 14.1 and we leave the details to the reader.

Theorem 14.2. *Suppose that $F, F_{,1}, F_{,2}, \ldots, F_{,N+1}$ are continuous on an open set A in $\mathbb{R}^{N+1}$ containing the point $P(x_1^0, x_2^0, \ldots, x_N^0, y^0)$. We use the notation $x = (x_1, x_2, \ldots, x_N)$, $x^0 = (x_1^0, x_2^0, \ldots, x_N^0)$ and suppose that*

$$F(x^0, y^0) = 0, \qquad F_{,N+1}(x^0, y^0) \neq 0.$$

(a) *Then there are positive numbers h and k which determine a cell R contained in A given by*

$$R = \{(x, y): |x_i - x_i^0| < h, \quad i = 1, 2, \ldots, N, |y - y^0| < k\},$$

such that for each x in the N-dimensional hypercube

$$I_N = \{x: |x_i - x_i^0| < h, \quad i = 1, 2, \ldots, N\}$$

there is a unique number y in the interval

$$J = \{y: |y - y^0| < k\}$$

which satisfies the equation $F(x, y) = 0$. That is, y is a function of x which may be written $y = f(x)$. The domain of f contains I_N and its range is in J.

(b) *The function f and its partial derivatives $f_{,1}, f_{,2}, \ldots, f_N$ are continuous on I_N.*

A special case of Theorem 14.1 is the Inverse function theorem which was established in Chapter 4 (Theorems 4.17 and 4.18). If f is a function from $\mathbb{R}^1$ to $\mathbb{R}^1$, denoted $y = f(x)$, we wish to decide when it is true that x may be expressed as a function of y. Set

$$F(x, y) = y - f(x) = 0$$

and, in order to apply Theorem 14.1, f' must be continuous and $F_{,1} = -f'(x) \neq 0$. We state the result in the following Corollary to Theorem 14.1.

Corollary (Inverse function theorem). *Suppose that f is defined on an open set A in $\mathbb{R}^1$ with values in $\mathbb{R}^1$. Also assume that f' is continuous on A and that $f(x_0) = y_0$, $f'(x_0) \neq 0$. Then there is an interval I containing y_0 such that the inverse function of f, denoted f^{-1}, exists on I and has a continuous derivative there. Furthermore, the derivative $(f^{-1})'$ is given by the formula*

$$(f^{-1}(y))' = \frac{1}{f'(x)}, \tag{14.4}$$

where $y = f(x)$.

Since $f^{-1}(f(x)) = x$, we can use the Chain rule to obtain (14.4). However, (14.4) is also a consequence of Formula (14.3), with $F(x, y) = y - f(x)$, and we find

$$(f^{-1}(y))' = -\frac{F_{,2}}{F_{,1}} = -\frac{1}{-f'(x)}.$$

Observe that in Theorems 4.17 and 4.18 the inverse mapping is one-to-one over the entire interval in which f' does not vanish.

EXAMPLE 1. Given the relation

$$F(x, y) = y^3 + 2x^2y - x^4 + 2x + 4y = 0, \tag{14.5}$$

show that this relation defines y as a function of x for all values of x in $\mathbb{R}^1$.

Solution. We have

$$F_{,2} = 3y^2 + 2x^2 + 4,$$

and so $F_{,2} > 0$ for all x, y. Hence for each fixed x, the function F is an increasing function of y. Furthermore, from (14.5) it follows that $F(x, y) \to -\infty$ as $y \to -\infty$ and $F(x, y) \to +\infty$ as $y \to +\infty$. Since F is continuous, for each fixed x there is exactly one value of y such that $F(x, y) = 0$. Applying Theorem 14.1, we conclude that there is a function f on $\mathbb{R}^1$ which is continuous and differentiable such that $F[x, f(x)] = 0$ for all x. □

EXAMPLE 2. Given the relation

$$F(x, y) = x^3 + y^3 - 6xy = 0, \tag{14.6}$$

find the values of x for which the relation defines y as a function of x (on some interval) and find the values of y for which the relation defines x as a function of y (on some interval).

Solution. The graph of the relation is shown in Figure 14.3. We see that

$$F_{,1} = 3x^2 - 6y, \qquad F_{,2} = 3y^2 - 6x,$$

and both partial derivatives vanish at (0, 0). We also observe that $F_{,2} = 0$ when $x = \frac{1}{2}y^2$, and substituting this value into the relation (14.6) we get $x = 2\sqrt[3]{4}$, $y = 2\sqrt[3]{2}$. The curve has a vertical tangent at this point, denoted P in Figure 14.3. Hence y is expressible as a function of x in a neighborhood of all points on the curve except P and the origin 0. Similarly $F_{,1} = 0$ yields the point Q with coordinates $(2\sqrt[3]{2}, 2\sqrt[3]{4})$. Then x is expressible as a function of y in a neighborhood of all points except Q and the origin 0. □

PROBLEMS

In each of Problems 1 through 4 show that the relation $F(x, y) = 0$ yields y as a function of x in an interval I about x_0 where $F(x_0, y_0) = 0$. Denote the function by f and compute f'.

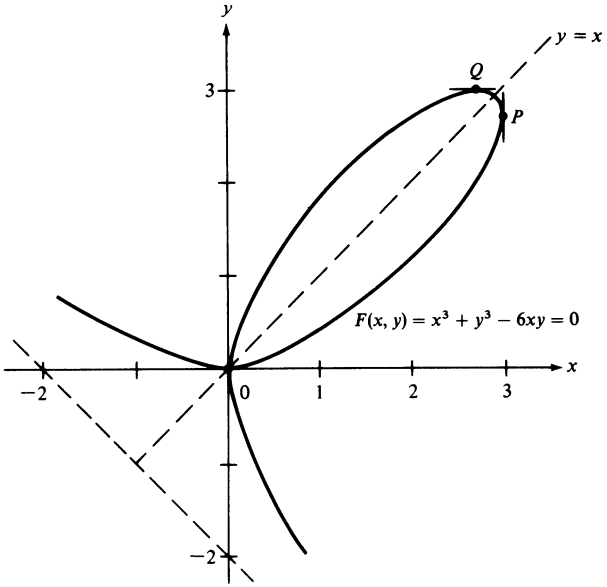

Figure 14.3

1. $F(x, y) \equiv y^3 + y - x^2 = 0$; $(x_0, y_0) = (0, 0)$.
2. $F(x, y) \equiv x^{2/3} + y^{2/3} - 4 = 0$; $(x_0, y_0) = (1, 3\sqrt{3})$.
3. $F(x, y) \equiv xy + 2 \ln x + 3 \ln y - 1 = 0$; $(x_0, y_0) = (1, 1)$.
4. $F(x, y) \equiv \sin x + 2 \cos y - \frac{1}{2} = 0$; $(x_0, y_0) = (\pi/6, 3\pi/2)$.
5. Give an example of a relation $F(x, y) = 0$ such that $F(x_0, y_0) = 0$ and $F_{,2}(x_0, y_0) = 0$ at a point $O = (x_0, y_0)$, and yet y is expressible as a function of x in an interval about x_0.

In each of Problems 6 through 9 show that the relation $F(x_1, x_2, y) = 0$ yields y as a function of (x_1, x_2) in a neighborhood of the given point $P(x_1^0, x_2^0, y^0)$. Denoting this function by f, compute $f_{,1}$ and $f_{,2}$ at P.

6. $F(x_1, x_2, y) \equiv x_1^3 + x_2^3 + y^3 - 3x_1x_2y - 4 = 0$; $P(x_1^0, x_2^0, y^0) = (1, 1, 2)$.
7. $F(x_1, x_2, y) \equiv e^y - y^2 - x_1^2 - x_2^2 = 0$; $P(x_1^0, x_2^0, y^0) = (1, 0, 0)$.
8. $F(x_1, x_2, y) \equiv x_1 + x_2 - y - \cos(x_1x_2y) = 0$; $P(x_1^0, x_2^0, y^0) = (0, 0, -1)$.
9. $F(x_1, x_2, y) \equiv x_1 + x_2 + y - e^{x_1x_2y} = 0$; $P(x_1^0, x_2^0, y^0) = (0, \frac{1}{2}, \frac{1}{2})$.
10. Prove Theorem 14.2.
11. Suppose that F is a function from $\mathbb{R}^2$ to $\mathbb{R}^1$ which we write $y = F(x_1, x_2)$. State hypotheses on F which imply that x_2 may be expressed as a function of x_1 and y (extension of the Inverse function theorem). Use Theorem 14.2.

12. Suppose that $F(x, y, z) = 0$ is such that the functions $z = f(x, y)$, $x = g(y, z)$, and $y = h(z, x)$ all exist by the Implicit function theorem. Show that

$$f_{,1} \cdot g_{,1} \cdot h_{,1} = -1.$$

This formula is frequently written

$$\frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial y} \cdot \frac{\partial y}{\partial z} = -1.$$

13. Find an example of a relation $F(x_1, x_2, y) = 0$ and a point $P(x_1^0, x_2^0, y^0)$ such that P satisfies the relation, and $F_{,1}(x_1^0, x_2^0, y^0) = F_{,2}(x_1^0, x_2^0, y^0) = F_{,3}(x_1^0, x_2^0, y^0) = 0$, yet y is a function of (x_1, x_2) in a neighborhood of P.

14. Suppose that the Implicit function theorem applies to $F(x, y) = 0$ so that $y = f(x)$. Find a formula for f'' in terms of F and its partial derivatives.

15. Suppose that the Implicit function theorem applies to $F(x_1, x_2, y) = 0$ so that $y = f(x_1, x_2)$. Find formulas for $f_{,1,1}$; $f_{,1,2}$; $f_{,2,2}$ in terms of F and its partial derivatives.

14.2. The Implicit Function Theorem for Systems

We shall establish an extension of the Implicit function theorem of Section 14.1 to systems of equations which define functions implicitly. A vector x in $\mathbb{R}^m$ has components denoted $(x_1, x_2, \ldots, x_m)$ and a vector y in $\mathbb{R}^n$ will have its components denoted by $(y_1, y_2, \ldots, y_n)$. An element in $\mathbb{R}^{m+n}$ will be written (x, y). We consider vector functions from $\mathbb{R}^{m+n}$ to $\mathbb{R}^n$ and write $F(x, y)$ for such a function. That is, F will have components

$$F^1(x, y), \qquad F^2(x, y), \ldots, F^n(x, y)$$

with each F^i a function from $\mathbb{R}^{m+n}$ to $\mathbb{R}^1$.

In order to establish the Implicit function theorem for systems we need several facts from linear algebra and a number of useful inequalities. We suppose the reader is familiar with the elements of linear algebra and in the next three lemmas we establish the needed inequalities.

Definition. Let A be an $m \times n$ matrix with elements

$$\begin{pmatrix} a_1^1 & a_2^1 & \cdots & a_n^1 \\ \vdots & \vdots & & \vdots \\ a_1^m & a_2^m & \cdots & a_n^m \end{pmatrix}.$$

The **norm** *of* A, written $|A|$, is defined by

$$|A| = \left[\sum_{i=1}^{m} \sum_{j=1}^{n} (a_j^i)^2 \right]^{1/2}.$$

Observe that for a vector, i.e., a $1 \times n$ matrix, the norm is the Euclidean length of the vector.

Lemma 14.1. *Let A be an $m \times n$ matrix, and suppose that $\zeta = (\zeta^1, \zeta^2, \dots, \zeta^n)$ is a column vector (that is, an $n \times 1$ matrix) with n components and that $\eta = (\eta^1, \eta^2, \dots, \eta^m)$ is a column vector with m components such that*

$$\eta = A\zeta,$$

or equivalently

$$\eta^i = \sum_{j=1}^{n} a_j^i \zeta^j, \qquad i = 1, 2, \dots, m. \tag{14.7}$$

Then

$$|\eta| \leqslant |A||\zeta|. \tag{14.8}$$

Proof. For fixed i in (14.7) we square both sides and apply the Schwarz inequality (Section 6.1), getting

$$(\eta^i)^2 \leqslant \sum_{j=1}^{n} (a_j^i)^2 \sum_{j=1}^{n} (\zeta^j)^2.$$

Then (14.8) follows by summing on i and taking the square root. □

The next lemma shows that with the above norm for matrices (and vectors) we can obtain an inequality for the estimation of integrals which resembles the customary one for absolute values.

Lemma 14.2. *Let $b: \mathbb{R}^m \to \mathbb{R}^n$ be a continuous vector function on a bounded, closed figure H in $\mathbb{R}^m$. Suppose that ζ is the $n \times 1$ column vector defined by*

$$\zeta = \int_H b \, dV_m.$$

That is,

$$\zeta^i = \int_H b^i \, dV_m, \qquad i = 1, 2, \dots, n,$$

where $b^i: \mathbb{R}^m \to \mathbb{R}^1$, $i = 1, 2, \dots, n$ are the components of b. Then

$$|\zeta| \leqslant \int_H |b| \, dV_m.$$

Proof. Define $\lambda_i = \zeta^i/|\zeta|$. Then multiplying by ζ^i and summing on i, we have $\sum_{i=1}^{n} \lambda_i \zeta^i = |\zeta|$. Therefore

$$|\zeta| = \sum_{i=1}^{n} \lambda_i \zeta^i = \sum_{i=1}^{n} \lambda_i \int_H b^i \, dV_m = \int_H \sum_{i=1}^{n} \lambda_i b^i \, dV_m.$$

We apply Lemma 14.1 and note that since $|\lambda| = 1$ we obtain

$$|\zeta| \leqslant \int_H |\lambda||b| \, dV_m = \int_H |b| \, dV_m.$$ □

Definitions. Let G be an open set in $\mathbb{R}^{m+n}$ and suppose that $F: G \to \mathbb{R}^n$ is a vector function $F(x, y)$ with continuous first partial derivatives. We *define the* $n \times m$ *and the* $n \times n$ *matrices* $\nabla_x F$ and $\nabla_y F$ by the formulas

$$\nabla_x F = \begin{pmatrix} \dfrac{\partial}{\partial x_1} F^1 & \cdots & \dfrac{\partial}{\partial x_m} F^1 \\ \vdots & & \vdots \\ \dfrac{\partial}{\partial x_1} F^n & \cdots & \dfrac{\partial}{\partial x_m} F^n \end{pmatrix}, \qquad \nabla_y F = \begin{pmatrix} \dfrac{\partial}{\partial y_1} F^1 & \cdots & \dfrac{\partial}{\partial y_n} F^1 \\ \vdots & & \vdots \\ \dfrac{\partial}{\partial y_1} F^n & \cdots & \dfrac{\partial}{\partial y_n} F^n \end{pmatrix}.$$

The Fixed point theorem of Chapter 13 will be used to establish the Implicit function theorem for systems. We note that in proving this theorem for a single equation we made essential use of the Mean-value theorem. The next lemma provides an appropriate generalization to systems of the Mean-value theorem.

Lemma 14.3. *Let G be an open set in $\mathbb{R}^{m+n}$ and $F: G \to \mathbb{R}^n$ a vector function with continuous first partial derivatives. Suppose that the straight line segment L joining $(\bar{x}, \bar{y})$ and $(\bar{\bar{x}}, \bar{\bar{y}})$ is in G and that there are two positive constants M_1, M_2 such that*

$$|\nabla_x F| \leqslant M_1 \quad \textit{and} \quad |\nabla_y F| \leqslant M_2$$

for all points (x, y) on the segment L. Then

$$|F(\bar{\bar{x}}, \bar{\bar{y}}) - F(\bar{x}, \bar{y})| \leqslant M_1 \cdot |\bar{\bar{x}} - \bar{x}| + M_2 \cdot |\bar{\bar{y}} - \bar{y}|.$$

Proof. Any point on the segment joining $(\bar{x}, \bar{y})$ to $(\bar{\bar{x}}, \bar{\bar{y}})$ has coordinates $(\bar{x} + t(\bar{\bar{x}} - \bar{x}), \bar{y} + t(\bar{\bar{y}} - \bar{y}))$ for $0 \leqslant t \leqslant 1$. We define the vector function

$$f(t) = F(\bar{x} + t(\bar{\bar{x}} - \bar{x}), \bar{y} + t(\bar{\bar{y}} - \bar{y}))$$

and use the simple fact that

$$f(1) - f(0) = \int_0^1 f'(t)\, dt.$$

Since $f(1) = F(\bar{\bar{x}}, \bar{\bar{y}})$, $f(0) = F(\bar{x}, \bar{y})$, it follows that

$$F(\bar{\bar{x}}, \bar{\bar{y}}) - F(\bar{x}, \bar{y}) = \int_0^1 \frac{d}{dt} F(\bar{x} + t(\bar{\bar{x}} - \bar{x}), \bar{y} + t(\bar{\bar{y}} - \bar{y}))\, dt.$$

Carrying out the differentiation with respect to t, and using the Chain rule, we find for each component F^i,

$$F^i(\bar{\bar{x}}, \bar{\bar{y}}) - F^i(\bar{x}, \bar{y})$$
$$= \int_0^1 \left\{ \sum_{j=1}^m \frac{\partial}{\partial x_j}(F^i)(\bar{\bar{x}}_j - \bar{x}_j) + \sum_{k=1}^n \frac{\partial}{\partial y_k}(F^i)(\bar{\bar{y}}_k - \bar{y}_k) \right\} dt.$$

In matrix notation we write

$$F(\bar{\bar{x}}, \bar{\bar{y}}) - F(\bar{x}, \bar{y}) = \int_0^1 [\nabla_x F \cdot (\bar{\bar{x}} - \bar{x}) + \nabla_y F \cdot (\bar{\bar{y}} - \bar{y})]\, dt.$$

From Lemma 14.2, it is clear that

$$|F(\bar{\bar{x}}, \bar{\bar{y}}) - F(\bar{x}, \bar{y})| \leqslant \int_0^1 [|\nabla_x F| \cdot |\bar{\bar{x}} - \bar{x}| + |\nabla_y F| \cdot |\bar{\bar{y}} - \bar{y}|]\, dt$$
$$\leqslant M_1 \cdot |\bar{\bar{x}} - \bar{x}| + M_2 \cdot |\bar{\bar{y}} - \bar{y}|. \qquad \square$$

For later use we next prove a simple proposition on nonsingular linear transformations.

Lemma 14.4. *Let B be an $n \times n$ matrix and suppose that $|B| < 1$. Define $A = I - B$ where I is the $n \times n$ identity matrix. Then A is nonsingular.*

Proof. Consider the mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ given by $y = Ax$, where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$. We show that the mapping is 1-1 thereby implying that A is nonsingular. Let $x_1, x_2 \in \mathbb{R}^n$; we have

$$Ax_1 - Ax_2 = (x_1 - x_2) - (Bx_1 - Bx_2)$$

and

$$|Bx_1 - Bx_2| = |B \cdot (x_1 - x_2)| \leqslant |B| \cdot |x_1 - x_2|.$$

Therefore

$$|Ax_1 - Ax_2| \geqslant |x_1 - x_2| - |Bx_1 - Bx_2| \geqslant |x_1 - x_2| - |B| \cdot |x_1 - x_2|$$
$$\geqslant |x_1 - x_2|(1 - |B|).$$

We conclude that if $x_1 \neq x_2$ then $Ax_1 \neq Ax_2$ and so the mapping is one-to-one. $\square$

The next lemma, a special case of the Implicit function theorem for systems, contains the principal ingredients for the proof of the main theorem. We establish the result for functions $F\colon \mathbb{R}^{m+n} \to \mathbb{R}^n$ which have the form

$$F(x, y) = y - Cx - \psi(x, y)$$

where C is a constant $n \times m$ matrix and ψ is such that it and its first partial derivatives vanish at the origin. Note the relation of this form of F with the second proof of the Implicit Function theorem for a single equation given in Theorem 14.1. Although the proof is lengthy, the reader will see that with the aid of the fixed point theorem of Chapter 13 and Lemma 14.3 the arguments proceed in a straightforward manner.

Lemma 14.5. *Let G be an open set in $\mathbb{R}^{m+n}$ which contains the origin. Suppose that $\psi\colon G \to \mathbb{R}^n$ is a continuous function with continuous first partial derivatives* *in G and that*

$$\psi(0, 0) = 0, \qquad \nabla_x\psi(0, 0) = 0, \qquad \nabla_y\psi(0, 0) = 0. \tag{14.9}$$

Suppose that C is a constant $n \times m$ matrix, and define the function $F\colon G \to \mathbb{R}^n$ by the formula

$$F(x, y) = y - Cx - \psi(x, y).$$

For any positive numbers r and s, denoted by $B_m(0, r)$ and $B_n(0, s)$ the balls in $\mathbb{R}^m$ and $\mathbb{R}^n$ with center at the origin and radii r and s, respectively. Then

(a) *There are (sufficiently small) positive numbers h and k with $B_m(0, h) \times B_n(0, k)$ in G and such that for each $x \in B_m(0, h)$ there is a unique element $y \in B_n(0, k)$ whereby*

$$F(x, y) = 0 \quad \text{or, equivalently,} \quad y = Cx + \psi(x, y).$$

(b) *If g denotes the function from $B_m(0, h)$ to $B_n(0, k)$ given by these ordered pairs (x, y), then g is continuous on $B_m(0, h)$ and all first partial derivatives of g are continuous on $B_m(0, h)$.*

PROOF

(a) Since G is open and ψ is continuous on G, there is a positive number k such that the closed set $B = \overline{B_m(0, k)} \times \overline{B_n(0, k)}$ is contained in G with ψ continuous on B. Also, because of (14.9) and the fact that the partial derivatives of ψ are continuous, k can be chosen so small that

$$|\nabla_x\psi| \leqslant \tfrac{1}{2}, \qquad |\nabla_y\psi| \leqslant \tfrac{1}{2} \quad \text{on } B.$$

We *fix* x in $B_m(0, k)$ and define the mapping T from $B_n(0, k)$ into $\mathbb{R}^n$ by the formula[1]

$$T(y) = Cx + \psi(x, y). \tag{14.10}$$

We apply Lemma 14.3 to $\psi(x, y)$, getting for $x \in B_m(0, k)$, $y \in B_n(0, k)$

$$|\psi(x, y)| = |\psi(x, y) - \psi(0, 0)| \leqslant \max|\nabla_x\psi|\cdot|x - 0| + \max|\nabla_y\psi|\cdot|y - 0|$$
$$\leqslant \tfrac{1}{2}|x| + \tfrac{1}{2}|y|.$$

Therefore, for $x \in B_m(0, k)$, $y \in B_n(0, k)$ it follows that

$$|T(y)| \leqslant |C|\cdot|x| + \tfrac{1}{2}|x| + \tfrac{1}{2}|y|. \tag{14.11}$$

Since C is a constant matrix there is a positive number M such that $|C| \leqslant M$. Now choose a positive number h which satisfies the inequality $h < k/(2M + 1)$. The mapping (14.10) will be restricted to those values of x in the ball $B_m(0, h)$. Then, from (14.10), for each fixed $x \in B_m(0, h)$ and $y \in B_n(0, k)$ we have

$$|T(y)| \leqslant (M + \tfrac{1}{2})h + \tfrac{1}{2}k < \tfrac{1}{2}k + \tfrac{1}{2}k = k;$$

[1] In the second proof of Theorem 14.1 we denoted this mapping by $T_x y$.

hence T maps $B_n(0, k)$ into itself. Furthermore, for $y_1, y_2 \in B_n(0, k)$ we find

$$|T(y_1) - T(y_2)| = |\psi(x, y_1) - \psi(x, y_2)| \leqslant \tfrac{1}{2}|y_1 - y_2|,$$

where Lemma 14.3 is used for the last inequality. Thus the mapping T is a contraction and the Fixed point theorem of Chapter 13 (Theorem 13.2) can be applied. For each *fixed* $x \in B_m(0, k)$ there is a unique $y \in B_n(0, k)$ such that

$$y = T(y) \quad \text{or} \quad y = Cx + \psi(x, y).$$

That is, y is a function of x which we denote by g. Writing $y = g(x)$, we observe that the equation $F(x, g(x)) = 0$ holds for all $x \in B_m(0, h)$.

(b) We show that g is continuous. Let $x_1, x_2 \in B_m(0, h)$ and $y_1, y_2 \in B_n(0, k)$ be such that $y_1 = g(x_1)$, $y_2 = g(x_2)$ or

$$y_1 = Cx_1 + \psi(x_1, y_1) \quad \text{and} \quad y_2 = Cx_2 + \psi(x_2, y_2).$$

Then

$$|y_2 - y_1| \leqslant |C(x_2 - x_1)| + |\psi(x_2, y_2) - \psi(x_1, y_1)|.$$

We use Lemma 14.3 for the last term on the right, getting

$$|y_2 - y_1| \leqslant M \cdot |x_2 - x_1| + \tfrac{1}{2}|x_2 - x_1| + \tfrac{1}{2}|y_2 - y_1|$$

or

$$|y_2 - y_1| \leqslant (2M + 1)|x_2 - x_1|.$$

Hence

$$|g(x_2) - g(x_1)| \leqslant (2M + 1)|x_2 - x_1|,$$

and g is continuous on $B_m(0, h)$.

We now show that the first partial derivatives of g exist and are continuous. Let the components of g be denoted by $g^1, g^2, \ldots, g^n$. We shall prove the result for a typical partial derivative $(\partial/\partial x_p)g^i$ where $1 \leqslant p \leqslant m$. In $\mathbb{R}^m$ let e^p denote the unit vector in the p-direction. That is, e^p has components $(e_1^p, e_2^p, \ldots, e_m^p)$ where $e_p^p = 1$ and $e_j^p = 0$ for $j \neq p$. Fix $\bar{x}$ in $B_m(0, h)$ and choose a positive number t_0 so small that the points $\bar{x} + te^p$ lie in $B_m(0, h)$ for all t such that $|t| \leqslant t_0$. Now set $x = \bar{x} + te^p$ and write

$$g(x) = Cx + \psi(x, g(x)).$$

The ith component of this equation reads

$$g^i(x) = \sum_{j=1}^{m} c_j^i x_j + \psi^i(x, g(x))$$

where the c_j^i are the components of the matrix C. Let Δg^i be defined by

$$\Delta g^i = g^i(\bar{x} + te^p) - g^i(\bar{x}).$$

Then from the Fundamental lemma on differentiation (Theorem 7.2), it follows that

$$\frac{\Delta g^i}{t} = c_p^i + \frac{\partial}{\partial x_p}\psi^i(x, g(x)) + \bar{\varepsilon}_p^i(te^p, \Delta g)$$
$$+ \sum_{k=1}^{n}\left[\frac{\partial}{\partial y_k}\psi^i(x, g(x)) + \bar{\bar{\varepsilon}}_k^i(te^p, \Delta g)\right]\frac{\Delta g^k}{t},$$

where $\bar{\varepsilon}_p^i(\rho, \sigma)$ and $\bar{\bar{\varepsilon}}_k^i(\rho, \sigma)$ are continuous at (0, 0) and vanish there. Taking the definition of the vector e^p into account, we can write the above expression in the form

$$\frac{\Delta g^i}{t} = \sum_{j=1}^{m} c_j^i e_j^p + \sum_{j=1}^{m}\frac{\partial}{\partial x_j}\psi^i(x, g(x))e_j^p + \sum_{j=1}^{m}\bar{\varepsilon}_j^i(te^p, \Delta g)e_j^p$$
$$+ \sum_{k=1}^{n}\left[\frac{\partial}{\partial y_k}\psi^i(x, g(x)) + \bar{\bar{\varepsilon}}_k^i(te^p, \Delta g)\right]\frac{\Delta g^k}{t} \tag{14.12}$$

where $\bar{\varepsilon}_j^i(\rho, \sigma)$, $j = 1, 2, \ldots, m$, are continuous at (0, 0) and vanish there. We define the matrices

$$A_1(t) = C + \nabla_x\psi(x, g(x)) + \bar{\varepsilon}(te^p, \Delta g),$$
$$A_2(t) = \nabla_y\psi(x, g(x)) + \bar{\bar{\varepsilon}}(te^p, \Delta g)$$

where the components of $\bar{\varepsilon}$ are $\bar{\varepsilon}_j^i$ and the components of $\bar{\bar{\varepsilon}}$ are $\bar{\bar{\varepsilon}}_k^i$, $i, k = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. Then (14.12) can be written as the single vector equation

$$\frac{\Delta g}{t} = A_1 e^p + A_2\frac{\Delta g}{t}. \tag{14.13}$$

Define $B = I - A_2$ where I is the $n \times n$ unit matrix. Then (14.13) becomes

$$B\frac{\Delta g}{t} = A_1 e^p. \tag{14.14}$$

According to (14.9) we have $|A_2(0)| \leqslant \frac{1}{2}$. Therefore, by Lemma 14.4 the matrix $B(0)$ is nonsingular. Since g is continuous on $B_m(0, h)$, we know that $\Delta g \to 0$ as $t \to 0$. Therefore the matrices $A_1(t)$, $A_2(t)$, and $B(t)$ are continuous at $t = 0$. Consequently $B(t)$ is nonsingular for t sufficiently close to zero. We allow t to tend to zero in (14.14) and conclude that the limit of $\Delta g/t$ exists; that is, $(\partial/\partial x_p)g^i$ exists for every i and every p. The formula

$$\lim_{t\to 0}\frac{\Delta g}{t} = B^{-1}(0)A_1(0)e^p$$

shows that the partial derivatives are continuous functions of x. □

Theorem 14.3 (Implicit function theorem for systems). *Let G be an open set in $\mathbb{R}^{m+n}$ containing the point $(\bar{x}, \bar{y})$. Suppose that $F\colon G \to \mathbb{R}^n$ is continuous and has continuous first partial derivatives in G. Suppose that*

$$F(\bar{x}, \bar{y}) = 0 \quad \textit{and} \quad \det \nabla_y F(\bar{x}, \bar{y}) \neq 0.$$

Then positive numbers h and k can be chosen so that: (a) *the direct product of the closed balls* $\overline{B_m(\bar{x}, h)}$ *and* $\overline{B_n(\bar{y}, k)}$ *with centers at* $\bar{x}$, $\bar{y}$ *and radii h and k, respectively, is in G; and* (b) *h and k are such that for each* $x \in B_m(\bar{x}, h)$ *there is a unique* $y \in B_n(\bar{y}, k)$ *satisfying* $F(x, y) = 0$. *If f is the function from* $B_m(\bar{x}, h)$ *to* $B_n(\bar{y}, k)$ *defined by these ordered pairs* (x, y), *then* $F(x, f(x)) = 0$; *furthermore, f and all its first partial derivatives are continuous on* $B_m(\bar{x}, h)$.

PROOF. We define the matrices

$$A = \nabla_x F(\bar{x}, \bar{y}), \qquad B = \nabla_y F(\bar{x}, \bar{y}),$$

and write F in the form

$$F(x, y) = A \cdot (x - \bar{x}) + B \cdot (y - \bar{y}) + \phi(x, y), \tag{14.15}$$

where ϕ is *defined*[2] by Equation (14.15). It is clear that ϕ has the properties

$$\phi(\bar{x}, \bar{y}) = 0, \qquad \nabla_x \phi(\bar{x}, \bar{y}) = 0, \qquad \nabla_y \phi(\bar{x}, \bar{y}) = 0.$$

By hypothesis, B is a nonsingular matrix. We multiply (14.15) by B^{-1}, getting

$$B^{-1}F = B^{-1}A \cdot (x - \bar{x}) + (y - \bar{y}) + B^{-1}\phi(x, y).$$

Now we may apply Lemma 14.5 with $B^{-1}F$ in place of F in that lemma, $x - \bar{x}$ in place of x; also, $y - \bar{y}$ in place of y, $-B^{-1}A$ in place of C, and $B^{-1}\phi$ in place of ψ. It is simple to verify that all the hypotheses of the lemma are fulfilled. The theorem follows for $B^{-1}F$. Since B^{-1} is a constant nonsingular matrix, the result holds for F. □

Remarks. The first partial derivatives of the implicitly defined function f may be found by a direct computation in terms of partial derivatives of F. To see this suppose that F has components $F^1, F^2, \ldots, F^n$ and that f has components $f^1, f^2, \ldots, f^n$. We write

$$F^i(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n) = 0, \tag{14.16}$$

where $y_i = f^i(x_1, x_2, \ldots, x_m)$. To find the partial derivatives of f^i, we take the derivative of F^i with respect to x_p in (14.16), getting (by the Chain rule)

$$\frac{\partial F^i}{\partial x_p} + \sum_{k=1}^{n} \frac{\partial F^i}{\partial y_k} \frac{\partial f^k}{\partial x_p} = 0, \qquad i = 1, 2, \ldots, n, p = 1, 2, \ldots, m. \tag{14.17}$$

Treating $\partial f^k/\partial x_p$ (for fixed p) as a set of n unknowns, we see that the above equations form an algebraic system of n equations in n unknowns in which, by hypothesis, the determinant of the coefficients does not vanish at $(\bar{x}, \bar{y})$. Therefore by Cramer's rule the system can always be solved uniquely.

[2] In the second proof of Theorem 14.1, the function φ is defined by: $F_{,2}(x_0, y_0)\psi(x, y)$.

EXAMPLE 1. Let $F(x, y)$ be a function from $\mathbb{R}^4$ to $\mathbb{R}^2$ given by

$$F^1(x_1, x_2, y_1, y_2) = x_1^2 - x_2^2 - y_1^3 + y_2^2 + 4,$$

$$F^2(x_1, x_2, y_1, y_2) = 2x_1x_2 + x_2^2 - 2y_1^2 + 3y_2^4 + 8.$$

Let $P(\bar{x}, \bar{y}) = (2, -1, 2, 1)$. It is clear that $F(\bar{x}, \bar{y}) = 0$. Verify that $\det \nabla_y F(\bar{x}, \bar{y}) \neq 0$ and find the first partial derivatives of the function $y = f(x)$ defined implicitly by F at the point P.

Solution. We have

$$\frac{\partial F^1}{\partial y_1} = -3y_1^2, \qquad \frac{\partial F^1}{\partial y_2} = 2y_2, \qquad \frac{\partial F^2}{\partial y_1} = -4y_1, \qquad \frac{\partial F^2}{\partial y_2} = 12y_2^3.$$

At P, we find

$$\det(\nabla_y F) = \frac{\partial F^1}{\partial y_1}\frac{\partial F^2}{\partial y_2} - \frac{\partial F^1}{\partial y_2}\frac{\partial F^2}{\partial y_1} = -128.$$

Also,

$$\frac{\partial F^1}{\partial x_1} = 2x_1, \qquad \frac{\partial F^1}{\partial x_2} = -2x_2, \qquad \frac{\partial F^2}{\partial x_1} = 2x_2, \qquad \frac{\partial F^2}{\partial x_2} = 2x_1 + 2x_2.$$

Substituting the partial derivatives evaluated at P in (14.17) and solving the resulting systems of two equations in two unknowns first with $p = 1$ and then with $p = 2$, we get

$$\frac{\partial f^1}{\partial x_1} = \frac{13}{32}, \qquad \frac{\partial f^2}{\partial x_1} = \frac{7}{16}, \qquad \frac{\partial f^1}{\partial x_2} = \frac{5}{32}, \qquad \frac{\partial f^2}{\partial x_2} = -\frac{1}{16}. \qquad \square$$

EXAMPLE 2. Given $F\colon \mathbb{R}^5 \to \mathbb{R}^3$ defined according to the formulas

$$F^1(x_1, x_2, y_1, y_2, y_3) = x_1^2 + 2x_2^2 - 3y_1^2 + 4y_1y_2 - y_2^2 + y_3^3,$$

$$F^2(x_1, x_2, y_1, y_2, y_3) = x_1 + 3x_2 - 4x_1x_2 + 4y_1^2 - 2y_2^2 + y_3^2,$$

$$F^3(x_1, x_2, y_1, y_2, y_3) = x_1^3 - x_2^3 + 4y_1^2 + 2y_2 - 3y_3^2.$$

Assume that $P(x, y)$ is a point where $F(x, y) = 0$ and $\nabla_y F$ is nonsingular. Denoting the implicit function by f, determine $\partial f^i/\partial x_j$ at P.

Solution. According to (14.17) a straightforward computation yields

$$(-6y_1 + 4y_2)\frac{\partial f^1}{\partial x_1} + (4y_1 - 2y_2)\frac{\partial f^2}{\partial x_1} + 3y_3^2\frac{\partial f^3}{\partial x_1} = -2x_1,$$

$$8y_1\frac{\partial f^1}{\partial x_1} - 4y_2\frac{\partial f^2}{\partial x_1} + 2y_3\frac{\partial f^3}{\partial x_1} = 4x_2 - 1,$$

$$8y_1\frac{\partial f^1}{\partial x_1} + 2\frac{\partial f^2}{\partial x_1} - 6y_3\frac{\partial f^3}{\partial x_1} = -3x_1^2.$$

We solve this linear system of three equations in three unknowns by Cramer's rule and obtain expressions for $\partial f^1/\partial x_1$, $\partial f^2/\partial x_1$, $\partial f^3/\partial x_1$. To find the partial derivatives of f with respect to x_2 we repeat the entire procedure, obtaining a similar linear system which can be solved by Cramer's rule. We leave the details to the reader. □

Definition. Let G be an open set in $\mathbb{R}^m$ and suppose that $F: G \to \mathbb{R}^n$ is a given vector function. The function F *is of* **class C^k on** G, where k is a nonnegative integer if and only if F and all its partial derivatives up to and including those of order k are continuous on G.

The Inverse function theorem which is a Corollary to Theorem 14.1 has a natural generalization for vector functions.

Theorem 14.4 (Inverse function theorem). *Let G be an open set in $\mathbb{R}^m$ containing the point $\bar{x}$. Suppose that $f: G \to \mathbb{R}^m$ is a function of class C^1 and that*

$$\bar{y} = f(\bar{x}), \qquad \det \nabla_x f(\bar{x}) \neq 0.$$

Then there are positive numbers h and k such that the ball $B_m(\bar{x}, k)$ is in G and for each $y \in B_m(\bar{y}, h)$ there is a unique point $x \in B_m(\bar{x}, k)$ with $f(x) = y$. If g is defined to be the inverse function determined by the ordered pairs (y, x) with the domain of g consisting of $B_m(\bar{y}, h)$ and range of g in $B_m(\bar{x}, k)$, then g is a function of class C^1. Furthermore, $f[g(y)] = y$ for $y \in B_m(\bar{y}, h)$.

Proof. This theorem is a corollary of Theorem 14.3 in which

$$F(y, x) = y - f(x).$$

□

Remarks. The Inverse function theorem for functions of one variable (Corollary to Theorem 14.1) has the property that the function is one-to-one over the entire domain in which the derivative does not vanish. In Theorem 14.4, the condition $\det \nabla_x f \neq 0$ does not guarantee that the inverse (vector) function will be one-to-one over its domain. To see this consider the function $f: \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f^1 = x_1^2 - x_2^2, \qquad f^2 = 2x_1 x_2, \tag{14.18}$$

with domain the annular ring $G \equiv \{(x_1, x_2): r_1 < (x_1^2 + x_2^2)^{1/2} < r_2\}$ where r_1, r_2 are positive numbers. A computation shows that

$$\nabla_x f = \begin{pmatrix} 2x_1 & -2x_2 \\ 2x_2 & 2x_1 \end{pmatrix},$$

and so $\det \nabla_x f = 4(x_1^2 + x_2^2)$, which is positive in G. However, setting $y = f(x)$, we see from (14.18) that there are two distinct values of x for each value of y. The inverse relation is a function in a sufficiently small ball of G, but if one considers the entire ring G there are two distinct values of $x = (x_1, x_2)$ in G which correspond to a given value of $y = (y_1, y_2)$.

Problems

In each of Problems 1 through 4 a function F and a point P are given. Verify that the Implicit function theorem is applicable. Denoting the implicitly defined function by f, find the values of all the first partial derivatives of f at P.

1. $F = (F^1, F^2)$, $P = (0, 0, 0, 0)$ where $F^1 = 2x_1 - 3x_2 + y_1 - y_2$, and $F^2 = x_1 + 2x_2 + y_1 + 2y_2$.
2. $F = (F^1, F^2)$, $P = (0, 0, 0, 0)$ where $F^1 = 2x_1 - x_2 + 2y_1 - y_2$, and $F^2 = 3x_1 + 2x_2 + y_1 + y_2$.
3. $F = (F^1, F^2)$, $P = (3, -1, 2, 1)$ where $F^1 = x_1 - 2x_2 + y_1 + y_2 - 8$, and $F^2 = x_1^2 - 2x_2^2 - y_1^2 + y_2^2 - 4$.
4. $F = (F^1, F^2)$, $P = (2, 1, -1, 2)$ where $F^1 = x_1^2 - x_2^2 + y_1y_2 - y_2^2 + 3$, and $F^2 = x_1 + x_2^2 + y_1^2 + y_1y_2 - 2$.
5. Suppose that $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $F: \mathbb{R}^4 \to \mathbb{R}^2$ are such that $F(x, y) = 0$ and the Implicit function theorem is applicable for all (x, y). Denoting the implicitly defined function by f, find a formula in terms of the first partial derivatives of F for $\partial f^1/\partial x_1$, $\partial f^1/\partial x_2$, $\partial f^2/\partial x_1$, $\partial f^2/\partial x_2$.
6. Suppose that $F(x, y) = 0$ where $x = (x_1, \ldots, x_m)$ and $y = (y_1, y_2)$ and that the Implicit function theorem is applicable. Denoting the implicitly defined function by f, find $\partial f^i/\partial x_j$, $i = 1, 2$, $j = 1, 2, \ldots, m$, in terms of the partial derivatives of F.
7. Complete Example 2.
8. Given $F = (F^1, F^2)$ where $F: \mathbb{R}^2 \to \mathbb{R}^2$ and $F^1 = e^{2x+y}$, $F^2 = (4x^2 + 4xy + y^2 + 6x + 3y)^{2/3}$. Show that there is no value of x for which the Implicit function theorem is applicable. Find a relation between F^1 and F^2.

In each of Problems 9 through 12 a vector function $f: \mathbb{R}^2 \to \mathbb{R}^2$ is given. Verify that the Inverse function theorem is applicable and find the inverse function g.

9. $y_1 = x_1$, $y_2 = x_1^2 + x_2$.
10. $y_1 = 2x_1 - 3x_2$, $y_2 = x_1 + 2x_2$.
11. $y_1 = x_1/(1 + x_1 + x_2)$, $y_2 = x_2/(1 + x_1 + x_2)$, $x_1 + x_2 > -1$.
12. $y_1 = x_1 \cos(\pi x_2/2)$, $y_2 = x_1 \sin(\pi x_2/2)$, $x_1 > 0$, $-1 < x_2 < 1$.
13. Given the function $f: \mathbb{R}^3 \to \mathbb{R}^3$ where $f^1 = e^{x_2} \cos x_1$, $f^2 = e^{x_2} \sin x_1$, and $f^3 = 2 - \cos x_3$. Find the points $P(x_1, x_2, x_3)$ where the Inverse function theorem holds.
14. Given the function $f: \mathbb{R}^2 \to \mathbb{R}^2$ and suppose that the Inverse function theorem applies. We write $x = g(y)$ for the inverse function. Find formulas for $\partial g^i/\partial y_j$, $i, j = 1, 2$ in terms of partial derivatives of f^1 and f^2. Also find a formula for $\partial^2 g^1/\partial y_2^2$.
15. Given $F: \mathbb{R}^4 \to \mathbb{R}^2$ and suppose that $F(x, y) = 0$ for all $x = (x_1, x_2)$ and $y =$

(y_1, y_2). State conditions which guarantee that the equation

$$\frac{\partial y_1}{\partial x_1}\frac{\partial x_2}{\partial y_1} + \frac{\partial y_2}{\partial x_1}\frac{\partial x_2}{\partial y_2} = 0$$

holds.

14.3. Change of Variables in a Multiple Integral

For functions of one variable, an integral of the form

$$\int f(x)\,dx$$

can be transformed into

$$\int f[g(u)]g'(u)\,du$$

by the "change of variable" $x = g(u)$, $dx = g'(u)\,du$. Such transformations are useful in the actual evaluation of many integrals. The corresponding result for multiple integrals is more complicated and, in order to establish the appropriate formula for such a change of variables, we employ several results in linear algebra. In this section we assume the reader is familiar with the basic facts concerning matrices and linear transformations.

Definition. Let G be an open set in $\mathbb{R}^m$ and let $f\colon G \to \mathbb{R}^m$ be a C^1 function. That is, f has components $f^1, f^2, \ldots, f^m$ and $f^i\colon G \to \mathbb{R}^1$ are C^1 functions for $i = 1, 2, \ldots, m$. The **Jacobian** of f is the $m \times m$ matrix having the first partial derivative $f^i_{,j}$ as the entry in the ith row and jth column, $i, j = 1, 2, \ldots, m$. We also use the terms **Jacobian matrix** and **gradient**, and we denote this matrix by ∇f.

In the next theorem we restate for vector functions the Fundamental lemma of differentiation (Part (a)) and the Chain rule (Part (b)). In Part (c) we give an extension to vector functions of Equation (14.4), the formula for the derivative of the inverse of a function.

Theorem 14.5. *Let G and G_1 be open sets in $\mathbb{R}^m$ with $\bar{x}$ a point in G. Let $f\colon G \to G_1$ be a C^1 function and denote $f = (f^1, f^2, \ldots, f^m)$.*

(a) *We have the formula* (*Fundamental Lemma of Differentiation*)

$$\begin{aligned} f^i(\bar{x} + h) - f^i(\bar{x}) &= \nabla f^i(\bar{x})h + \varepsilon^i(h) \\ &= \sum_{j=1}^{m} f^i_{,j}h_j + \varepsilon^i(h), \qquad i = 1, 2, \ldots, m, \end{aligned} \tag{14.19}$$

where $h = (h_1, h_2, \ldots, h_m)$ *and* $\varepsilon(h) = (\varepsilon^1(h), \ldots, \varepsilon^m(h))$ *are vectors, and* $\lim_{|h|\to 0} \varepsilon^i(h)/|h| = 0$.

(b) *Let* $g\colon G_1 \to \mathbb{R}^m$ *be of class* C^1 *and define* $F(x) = g[f(x)]$ *for* $x \in G$. *Then we have the Chain Rule*:

$$\nabla F(x) = \nabla g[f(x)] \cdot \nabla f(x). \tag{14.20}$$

(c) *Suppose that* f *is one-to-one with* $\det \nabla f(x) \neq 0$ *on* G. *Then the image* $f(G) = G_0$ *is open and the inverse function* $g_1 = f^{-1}$ *is one-to-one on* G_0 *and of class* C^1. *Furthermore,*

$$\nabla g_1[f(x)] = [\nabla f(x)]^{-1} \quad \textit{with} \quad \det([\nabla f(x)]^{-1}) \neq 0 \quad \textit{for} \quad x \in G \tag{14.21}$$

or

$$\nabla g_1(u) = \{\nabla f[f^{-1}(u)]\}^{-1} \neq 0, \qquad u \in G_0.$$

PROOF

(a) Formula (14.19) follows directly from the Fundamental lemma of differentiation for functions in $\mathbb{R}^m$ as given in Theorem 7.2.

(b) Formula (14.20) is a consequence of the Chain rule for partial derivatives as stated in Theorem 7.3. Each component of ∇F may be written (according to Theorem 7.3)

$$F^i_{,j}(x) = \sum_{j=1}^{m} g^i_{,k}[f(x)] \cdot f^k_{,j}(x),$$

which is (14.20) precisely.

(c) Since f is one-to-one, it is clear that f^{-1} is a function. Let $\bar{y} \in G_0$ where G_0 is the image of G and suppose $f(\bar{x}) = \bar{y}$. From the Inverse function theorem, which is applicable since $\nabla f(\bar{x}) \neq 0$, there are positive numbers h and k such that the ball $B(\bar{x}, k)$ is in G and also such that for each $y \in B(\bar{y}, h)$ there is a unique $x \in B(\bar{x}, k)$ with the property that $f(x) = y$. We define $g_1(y)$ to be the function given by the pairs (y, x). Then g_1 is of class C^1 on $B(\bar{y}, h)$ and the domain of g_1 contains $B(\bar{y}, h)$. Hence for each $\bar{y}$ in G_0, there is a ball with $\bar{y}$ as center which is also in G_0. We conclude that G_0 is open. Formula (14.21) follows from (14.20) and the Inverse function theorem. □

In establishing the change of variables formula we shall see that an essential step in the proof is the reduction of any C^1 function f into the composition of a sequence of functions which have a somewhat simpler character. This process can be carried out whenever the Jacobian of f does not vanish.

Definition. Let $(i_1, i_2, \ldots, i_m)$ be a permutation of the numbers $(1, 2, \ldots, m)$. A linear transformation τ from $\mathbb{R}^m$ into $\mathbb{R}^m$ is **simple** if τ has the form

$$\tau(x_1, x_2, \ldots, x_m) = (\pm x_{i_1}, \pm x_{i_2}, \ldots, \pm x_{i_m}).$$

The next lemma is an immediate consequence of the above definition.

Lemma 14.6. *The product of simple transformations is simple and the inverse of a simple transformation is simple.*

If f_1 and f_2 are functions on $\mathbb{R}^m$ to $\mathbb{R}^m$ such that the range of f_2 is in the domain of f_1, we use the notation $f_1 \circ f_2$ for the composition $f_1[f_2(x)]$ of the two functions.

The next lemma gives the precise reduction of a function on $\mathbb{R}^m$ as the composition of functions each of which has an essentially simpler character.

Lemma 14.7. *Let G be an open set in $\mathbb{R}^m$, $\bar{x} \in G$, and let $f: G \to \mathbb{R}^m$ be a C^1 function with* det $\nabla f(\bar{x}) \neq 0$. *Then there is an open subset G_1 of G containing $\bar{x}$ such that f can be written on G_1 as the composition of $m + 1$ functions*

$$f = g_{m+1} \circ g_m \circ \cdots \circ g_1. \tag{14.22}$$

The first m functions $g_1, g_2, \ldots, g_m$ are each defined on an open set G_i in $\mathbb{R}^m$ with range on an open set in $\mathbb{R}^m$ such that g_i: $G_i \to G_{i+1}, i = 1, 2, \ldots, m$. Moreover, the components $(g_i^1, g_i^2, \ldots, g_i^m)$ of g_i have the form

$$g_i^j(x_1, x_2, \ldots, x_m) = x_j \quad \text{for} \quad j \neq i \quad \text{and} \quad g_i^i = \varphi^i(x_1, x_2, \ldots, x_m). \tag{14.23}$$

The functions φ^i are determined in terms of f and have the property that $\varphi^i_{,i} > 0$ on G_i. The function g_{m+1} is simple.

Proof. Since all the components except one in the definition of g_i given by (14.23) are coordinate functions, a straightforward computation shows that the determinant of the matrix ∇g_i, denoted det ∇g_i, is equal to $\varphi^i_{,i}$. We shall establish that $\varphi^i_{,i}$ is positive and so these determinants will all be positive.

Since the Jacobian $\nabla f(\bar{x})$ is nonsingular, there is a linear transformation τ_1 such that $\tau_1 \circ f$ has the property that all the principal minors of the Jacobian $\nabla(\tau_1 \circ f)$ are positive at $\bar{x}$. Define $f_0 = \tau_1 \circ f$ and denote the components of f_0 by $(f_0^1, f_0^2, \ldots, f_0^m)$. Next define m functions $h_1, h_2, \ldots, h_m$ as follows:

$$h_i \text{ has components } (f_0^1, f_0^2, \ldots, f_0^i, x_{i+1}, x_{i+2}, \ldots, x_m)$$

for $i = 1, 2, \ldots, m - 1$. We set $h_m = f_0$. Since all the principal minors of ∇f_0 are positive, it is not difficult to see that each $\nabla h_i(\bar{x})$ is nonsingular and, in fact, det $\nabla h_i(\bar{x}) > 0$ for each i. According to Part (c) of Theorem 14.5 and the manner in which the h_i are defined, for each i there is an open set H_i on which det $\nabla h_i(x) > 0$. Also, h_i is one-to-one from H_i onto an open set. Define

$$G_1 = H_1 \cap H_2 \cap \cdots \cap H_m.$$

Now, define sets $G_2, G_3, \ldots, G_{m+1}$ as follows:

$$G_{i+1} = h_i(G_1), \qquad i = 1, 2, \ldots, m.$$

Henceforth we restrict the domain of $h_1, \ldots, h_m$ to be G_1 without relabeling the functions. Define

$$g_1 = h_1, \qquad g_i = h_i \circ h_{i-1}^{-1}, \qquad i = 2, 3, \ldots, m. \tag{14.24}$$

To define g_{m+1} consider the function inverse to τ_1, denoted τ^{-1} which, like τ_1 is a linear function. Define

$$g_{m+1} = \tau^{-1} \quad \text{restricted to} \quad G_{m+1}.$$

We observe that each function g_i is a one-to-one mapping from G_i onto G_{i+1}, and that det $\nabla g_i(x) \neq 0$ on G_i. Also,

$$\begin{aligned} g_{m+1} \circ g_m \circ \cdots \circ g_2 \circ g_1 &= \tau^{-1} \circ h_m \circ h_{m-1}^{-1} \circ h_{m-1} \circ h_{m-2}^{-1} \circ \cdots \circ h_2 \circ h_1^{-1} \circ h_1 \\ &= \tau^{-1} \circ h_m \\ &= \tau^{-1} \circ \tau_1 \circ f = f, \end{aligned}$$

and so (14.22) holds.

Once we show that each function g_i has the form given by (14.23) the proof will be complete. Since

$$g_1 = h_1 = (f_0^1, x_2, \ldots, x_m)$$

it is clear that g_1 has the proper form. Now $g_2 = h_2 \circ h_1^{-1}$ and since $h_1^{-1} = (f_0^{-1}, x_2, \ldots, x_m)$, $h_2 = (f_0^1, f_0^2, x_3, \ldots, x_m)$, we see that

$$g_2 = (x_1, f_0^2, x_3, \ldots, x_m).$$

The argument for each g_i is similar. Since all the principal minors of ∇f_0 are positive, we know that $f_{0,i}^i > 0$ and, from the way we selected the vectors h_i and g_i we conclude that $\varphi_{,i}^i = f_{0,i}^i > 0$. □

In Lemma 14.7 we express an arbitrary C^1 function f as the composition of functions each of which is the identity in all components except one (plus a simple function). The one component which is not the identity, for example the ith, has the property that its partial derivative with respect to x_i is positive on G_i.

The next step (Lemma 14.9) establishes the change of variables formula for a typical function which appears in such a decomposition.

Lemma 14.8. *Let G be a set in $\mathbb{R}^m$ and $\varphi: G \to \mathbb{R}^1$ a bounded function such that $|\varphi(x) - \varphi(y)| \leqslant \varepsilon$ for all $x, y \in G$. Define $m = \inf\{\varphi(x): x \in G\}$ and $M = \sup\{\varphi(x): x \in G\}$. Then*

$$M - m \leqslant \varepsilon.$$

The proof is left to the reader.

Lemma 14.9. *Let G be an open set in $\mathbb{R}^m$ and suppose that $f: G \to \mathbb{R}^m$ is a one-to-one function of class C^1. We denote the components of f by $(u^1, u^2, \ldots, u^m)$ and let k be a fixed integer between 1 and m. Suppose the u^i have the form*

$$u^i(x_1, x_2, \ldots, x_m) = x_i, \quad \text{the } i\text{th coordinate in } \mathbb{R}^m \text{ for } i \neq k,$$

$$u^k = f^k(x) \quad \text{with} \quad f_{,k}^k(x) > 0 \quad \text{on} \quad G.$$

(a) *If F is a figure with $\bar{F} \subset G$, then the set $f(F)$ is a figure in $\mathbb{R}^m$.*
(b) *Denote $f(G)$ by G_1 and let $K: G_1 \to \mathbb{R}^1$ be uniformly continuous on G_1. Then the change of variables formula holds:*

$$\int_F K[f(x)]|J(x)|\,dV_m = \int_{f(F)} K(u)\,dV_m, \quad \textit{where} \quad J(x) = \det \nabla f(x). \quad (14.25)$$

PROOF

(a) Let R *be a closed cell in* G *with* $a_i \leqslant x_i \leqslant b_i$, $i = 1, 2, \ldots, m$ (see Section 8.1). Then the set $S = f(R)$ is given by

$$a_i \leqslant u^i \leqslant b_i, \qquad i \neq k,$$

and

$$f^k(x_1, \ldots, x_{k-1}, a_k, x_{k+1}, \ldots, x_m) \leqslant u^k \leqslant f^k(x_1, \ldots, x_{k-1}, b_k, x_{k+1}, \ldots, x_m).$$

We now employ the Corollary to Theorem 8.13 to conclude not only that S is a figure in $\mathbb{R}^m$ but that its volume, denoted $V(S) = V[f(R)]$, is given by

$$V(S) = \int_{a_k'}^{b_k'} [f^k(x_1, \ldots, x_{k-1}, b_k, x_{k+1}, \ldots, x_m)$$
$$- f^k(x_1, \ldots, x_{k-1}, a_k, x_{k+1}, \ldots, x_m)]\,dx_1 \ldots dx_{k-1} dx_{k+1} \ldots dx_m.$$

The symbols a_k', b_k' mean that the integration with respect to each variable x_i is between the limits a_i and b_i. Since the integrand above can be written as

$$\int_{a_k}^{b_k} f_{,k}^k\,dx_k,$$

we find

$$V(S) = \int_R f_{,k}^k(x)\,dV_m,$$

where dV_m is the usual element of volume in $\mathbb{R}^m$. From the way we defined f, a simple computation shows that $|J(x)| = f_{,k}^k(x)$, and we conclude that

$$V[f(R)] = \int_R |J(x)|\,dV_m.$$

Part (a) is now established when F is a cell. Next, let F be any figure such that $\bar{F} \subset G$. For any positive integer n we may cover F with hypercubes of side 2^{-n}, denoting by F_n^- the collection of inner hypercubes and by F_n^+ the collection of inner and boundary hypercubes. From the Lebesgue lemma (Theorem 3.16 and Theorem 6.27) which is valid in $\mathbb{R}^m$ it follows that there is a positive number ρ such that all members of F_n^+ are entirely in G and, in fact are at least at distance ρ from the boundary of G. Since no two hypercubes of F_n^+ have interior points in common, it follows that

$$V[f(F_n^+)] = \int_{F_n^+} |J(x)|\,dV,$$

and a similar formula holds for F_n^-. Denoting the inner and outer volume of $f(F)$ by $V^-[f(F)]$ and $V^+[f(F)]$, respectively, we find

$$\begin{aligned}\int_{F_n^-} |J(x)|\, dV = V[f(F_n^-)] &\leqslant V^-[f(F)] \\ &\leqslant V^+[f(F)] \leqslant V[f(F_n^+)] \\ &= \int_{F_n^+} |J(x)|\, dV. \end{aligned} \tag{14.26}$$

Since f is of class C^1, the function $|J(x)|$ is uniformly continuous on F_n^+ for all sufficiently large n and hence bounded by a constant which we denote by M. Therefore,

$$\int_{F-F_n^-} |J(x)|\, dV \leqslant MV(F - F_n^-), \qquad \int_{F_n^+-F} |J(x)|\, dV \leqslant MV(F_n^+ - F).$$

Since F is a figure, we let $n \to \infty$ and these integrals tend to zero. Employing this fact in (14.26) we conclude that $V^-[f(F)] = V^+[f(F)]$, and so $f(F)$ is a figure. Moreover,

$$V[f(F)] = \int_F |J(x)|\, dV,$$

that is, in addition to Part (a) we showed that Formula (14.25) holds in the special case $K(x) \equiv 1$.

(b) Let F be a figure such that $\bar{F} \subset G$. Since f and $|J|$ are continuous on $\bar{F}$, a closed bounded set, they are uniformly continuous on $\bar{F}$. Since, by hypothesis, $K[f(x)]$ is uniformly continuous on G_1, we see that the function $K[f(x)] \cdot |J(x)|$ is uniformly continuous on $\bar{F}$ and hence integrable on F. We shall establish Formula (14.25) by approximating each of the integrals in (14.25) by a Riemann sum and then by showing that the two Riemann sums are arbitrarily close if the subdivision is sufficiently fine. Let $\varepsilon > 0$ be given, and let Δ: $\{F_1, F_2, \ldots, F_n\}$ be a subdivision of F. Choose $\xi_i \in F_i$, $i = 1, 2, \ldots, n$. Then

$$\left|\sum_{i=1}^n K[f(\xi_i)]\,|J(\xi_i)|\, V_m(F_i) - \int_F K[f(x)]\,|J(x)|\, dV_m\right| < \frac{\varepsilon}{3}, \tag{14.27}$$

if the mesh $\|\Delta\|$ is sufficiently small, say less than some number δ. Similarly, if Δ_1: $\{F_1', F_2', \ldots, F_n'\}$ is a subdivision of $f(F)$ with $\|\Delta_1\| < \eta$ and with $\xi_i' \in F_i'$, $i = 1, 2, \ldots, n$, then for sufficiently small η, it follows that

$$\left|\sum_{i=1}^n K(\xi_i') V_m(F_i') - \int_{f(F)} K(u)\, dV_m\right| < \frac{\varepsilon}{3}. \tag{14.28}$$

Let $M = \sup_{u \in f(F)} |K(u)|$. Because of the uniform continuity of f and $|J|$, we may choose δ so small that for all $x', x'' \in F$ with $|x' - x''| < \delta$, we have

$$|f(x') - f(x'')| < \eta \quad \text{and} \quad \big||J(x')| - |J(x'')|\big| < \frac{\varepsilon}{3MV_m(F)}. \tag{14.29}$$

We now assume δ is chosen in this way and that δ is made smaller, if necessary, so that (14.27) holds. Select $\xi_i' = f(\xi_i)$ and $F_i' = f(F_i)$, $i = 1, 2, \ldots, n$. Then Δ_1: $\{F_1', \ldots, F_n'\}$ is a subdivision of $f(F)$, and from the first inequality in (14.29), we have $\|\Delta_1\| < \eta$. Thus (14.28) holds. Next denote by m_i and M_i the infimum and supremum of $|J(x)|$ on F_i, respectively. Then from the proof of Part (a) and the Mean-value theorem for integrals, it follows that

$$V_m(F_i') = \int_{F_i} |J(x)|\, dV_m = |J_i|\, V_m(F_i), \tag{14.30}$$

where $|J_i|$ is a number such that $m_i \leqslant |J_i| \leqslant M_i$. We also have $m_i \leqslant |J(\xi_i)| \leqslant M_i$ and so, by Lemma 14.8 and the second inequality in (14.29), we find

$$\big\||J_i| - |J(\xi_i)|\big\| \leqslant M_i - m_i \leqslant \frac{\varepsilon}{3MV_m(F)}. \tag{14.31}$$

We wish to estimate the difference of the Riemann sums

$$\left|\sum_{i=1}^{n} K(\xi_i')V_m(F_i') - \sum_{i=1}^{n} K[f(\xi_i)]\,|J(\xi_i)|\, V_m(F_i)\right|. \tag{14.32}$$

Using (14.30) and the fact that $\xi_i' = f(\xi_i)$, we obtain for (14.32)

$$\left|\sum_{i=1}^{n} K(\xi_i')[|J_i| - |J(\xi_i)|]\, V_m(F_i)\right|.$$

Inserting (14.31) into this expression, we find that

$$\left|\sum_{i=1}^{n} K(\xi_i')V_m(F_i') - \sum_{i=1}^{n} K[f(\xi_i)]\,|J(\xi_i)|\, V_m(F_i)\right| < \frac{\varepsilon}{3}. \tag{14.33}$$

Combining (14.27), (14.28), and (14.33), we conclude that

$$\left|\int_F K[f(x)]\,|J(x)|\, dV_m - \int_{f(F)} K(u)\, dV_m\right| < \varepsilon.$$

Since ε is arbitrary Formula (14.25) holds. □

Lemma 14.10. *Suppose that f: $G \to G_1$ is simple. Then the conclusions of Lemma 14.9 hold.*

Proof. If f is simple the image of any cell in G is a cell in G_1 (perhaps with the sides arranged in a different order). Also, for f simple, we have $|J(x)| = 1$, and $|f(x') - f(x'')| = |x' - x''|$ for any two points $x', x'' \in G$. The remaining details may be filled in by the reader. □

In Lemma 14.7 we showed how to express a function f as the composition of essentially simpler functions $g_1, g_2, \ldots, g_m$. Then, in Lemmas 14.9 and 14.10 we established the change of variables formula for these simpler functions. Now we show that the change of variables formula, (14.25), holds in general.

Theorem 14.6 (Change of variables formula). *Let G be an open set in $\mathbb{R}^m$ and suppose that $f\colon G \to \mathbb{R}^m$ is one-to-one and of class C^1 with* $\det \nabla f(x) \neq 0$ *on G. Let F be a closed bounded figure contained in G. Suppose that $K\colon f(F) \to \mathbb{R}^1$ is continuous on $f(F)$. Then $f(F)$ is a figure, $K[f(x)]$ is continuous on F, and*

$$\int_{f(F)} K(u)\,dV_m = \int_F K[f(x)]\cdot|J(x)|\,dV_m, \quad \textit{where} \quad J(x) = \det \nabla f(x). \qquad (14.34)$$

Proof. Let x_0 be any point of G. Then, according to Lemma 14.7, there is an open set G_1 with $x_0 \in G_1$, $G_1 \subset G$, and such that on G_1 we have $f = g_{m+1} \circ g_m \circ \cdots \circ g_1$ with the g_i satisfying all the conditions of Lemma 14.7. In Lemmas 14.9 and 14.10, we established the change of variables formula for each g_i.

Let F be a closed bounded figure in G_1 and suppose K is continuous on $f(F)$. The set $f(F)$ is given by

$$f(F) = g_{m+1} \circ g_m \circ \cdots \circ g_1(F).$$

Applying Lemma 14.10 to the simple mapping g_{m+1}, we see that the set $g_m \circ g_{m-1} \circ \cdots \circ g_1(F)$ is a figure. Define the function

$$K_1(u) = K[g_{m+1}(u)]\cdot|\det \nabla g_{m+1}(u)|.$$

Then K_1 is continuous on $g_m \circ g_{m-1} \circ \cdots \circ g_1(F)$ and

$$\begin{aligned}\int_{f(F)} K(u)\,dV_m &= \int_{g_{m+1}\circ\cdots\circ g_1(F)} K(u)\,dV_m \\ &= \int_{g_m\circ\cdots\circ g_1(F)} K_1(u)\,dV_m. \end{aligned} \qquad (14.35)$$

Next apply Lemma 14.9 to the mapping g_m. We define

$$K_2(u) = K_1[g_m(u)]\cdot|\det \nabla g_m(u)|,$$

and we observe that $g_{m-1} \circ g_{m-2} \circ \cdots \circ g_1(F)$ is a figure with K_2 continuous on this set. Therefore from Lemma 14.9, we have

$$\int_{f(F)} K(u)\,dV_m = \int_{g_m\circ\cdots\circ g_1(F)} K_1(u)\,dV_m = \int_{g_{m-1}\circ\cdots\circ g_1(F)} K_2(u)\,dV_m.$$

By substitution, we find

$$\begin{aligned}K_2(u) &= K_1[g_m(u)]\cdot|\det \nabla g_m(u)| \\ &= K[g_{m+1}(g_m(u))]\cdot|\det \nabla g_{m+1}[g_m(u)]|\cdot|\det \nabla g_m(u)|.\end{aligned}$$

Set $h_2(u) = g_{m+1} \circ g_m(u)$ and then the above formula becomes

$$K_2(u) = K[h_2(u)]\cdot|\det \nabla h_2(u)|,$$

where the Chain rule and the formula for the product of determinants have been used. We continue this process by defining $h_p(u) = g_{m+1} \circ g_m \circ \cdots \circ g_{m-p+2}$, $p = 2, 3, \ldots, m+1$, and $K_p(u) = K[h_p(u)]\cdot|\det \nabla h_p(u)|$. We arrive at the formula

$$\int_{g_{m+1}\circ\cdots\circ g_1(F)} K(u)\,dV_m = \int_{g_1(F)} K[h_m(u)]\cdot|\det \nabla h_m(u)|\,dV_m$$
$$= \int_F K[h_{m+1}(u)]\cdot|\det \nabla h_{m+1}(u)|\;dV_m$$
$$= \int_F K[f(u)]\cdot|\det \nabla f(u)|\,dV_m,$$

which is the desired result for a figure F in G_1.

To complete the proof, let F be any closed bounded figure in G and suppose that K is continuous on F. From the Lebesgue lemma (Theorems 3.16 and 6.27), there is a number ρ such that any ball $B(x, \rho)$ with center at a point of F lies in some open set G_1. We subdivide F into a finite number of figures $F_1, F_2, \ldots, F_s$ such that each F_i is contained in a single ball $B(x, \rho)$. For $i = 1, 2, \ldots, s$, we have

$$\int_{f(F_i)} K(u)\,dV_m = \int_{F_i} K[f(v)]\,|\det \nabla f(v)|\,dV_m.$$

The formula (14.34) follows by addition on i. □

EXAMPLE. Evaluate $\int_F x_1\,dV_2(x)$ where F is the region bounded by the curves $x_1 = -x_2^2$, $x_1 = 2x_2 - x_2^2$, and $x_1 = 2 - 2x_2 - x_2^2$ (see Figure 14.4(a)). Introduce new variables (u_1, u_2) by

$$f\colon x_1 = u_1 - \tfrac{1}{4}(u_1 + u_2)^2, \qquad x_2 = \tfrac{1}{2}(u_1 + u_2), \tag{14.36}$$

and use Theorem 14.6.

Solution. Figure 14.4 shows G, the image of F in the (u_1, u_2)-plane. Solving (14.36) for u_1, u_2 in terms of x_1, x_2, we get

$$u_1 = x_1 + x_2^2, \qquad u_2 = -x_1 + 2x_2 - x_2^2,$$

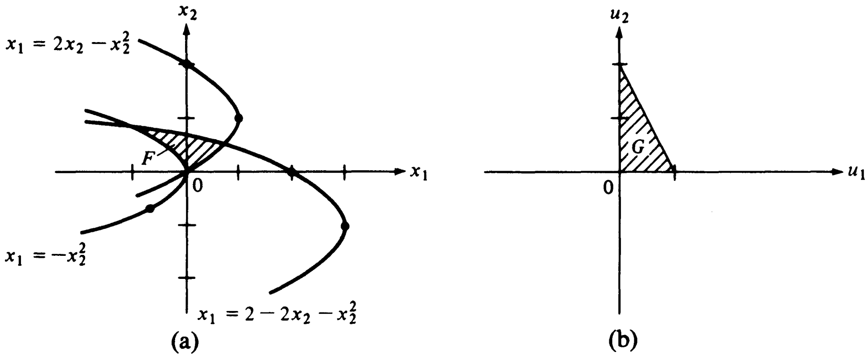

Figure 14.4. Changing variables from (x_1, x_2) to (u_1, u_2).

and so f is a one-to-one transformation of $\mathbb{R}^2$ onto itself. The equations of the bounding curves of G are

$$u_1 = 0, \qquad u_2 = 0, \qquad 2u_1 + u_2 = 2.$$

The Jacobian of f is

$$\det \nabla f = \begin{vmatrix} 1 - \frac{1}{2}(u_1 + u_2) & -\frac{1}{2}(u_1 + u_2) \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}.$$

Therefore

$$\int_F x_1\, dV_2(x) = \int_G \left[u_1 - \frac{1}{4}(u_1 + u_2)^2\right] \cdot \frac{1}{2}\, dV_2(u)$$
$$= \frac{1}{2}\int_0^1 \int_0^{2-2u_1} \left[u_1 - \frac{1}{4}(u_1 + u_2)^2\right] du_2\, du_1 = \frac{1}{48}. \qquad \square$$

Problems

In each of Problems 1 through 6 evaluate $\int_F K(x_1, x_2)\, dV_2(x)$, where F is bounded by the curves whose equations are given. Perform the integration by introducing variables u_1, u_2 as indicated. Draw a graph of F and the corresponding region in the u_1, u_2-plane. Find the inverse of each transformation.

1. $K(x_1, x_2) = x_1 x_2$. F is bounded by $x_2 = 3x_1$, $x_1 = 3x_2$, and $x_1 + x_2 = 4$. Mapping: $x_1 = 3u_1 + u_2$, $x_2 = u_1 + 3u_2$.
2. $K(x_1, x_2) = x_1 - x_2^2$. F is bounded by $x_2 = 2$, $x_1 = x_2^2 - x_2$, $x_1 = 2x_2 + x_2^2$. Mapping: $x_1 = 2u_1 - u_2 + (u_1 + u_2)^2$, $x_2 = u_1 + u_2$.
3. $K(x_1, x_2) = x_2$. F is bounded by $x_1 + x_2 - x_2^2 = 0$, $2x_1 + x_2 - 2x_2^2 = 1$, $x_1 - x_2^2 = 0$. Mapping: $x_1 = u_1 + (u_2 - u_1)^2$, $x_2 = u_2$.
4. $K(x_1, x_2) = (x_1^2 + x_2^2)^{-3}$. F is bounded by $x_1^2 + x_2^2 = 2x_1$, $x_1^2 + x_2^2 = 4x_1$, $x_1^2 + x_2^2 = 2x_2$, $x_1^2 + x_2^2 = 6x_2$. Mapping: $x_1 = u_1/(u_1^2 + u_2^2)$, $x_2 = u_2/(u_1^2 + u_2^2)$.
5. $K(x_1, x_2) = 4x_1 x_2$. F is bounded by $x_1 = x_2$, $x_1 = -x_2$, $(x_1 + x_2)^2 + x_1 - x_2 - 1 = 0$. Mapping: $x_1 = \frac{1}{2}(u_1 + u_2)$, $x_2 = \frac{1}{2}(-u_1 + u_2)$. Assume $x_1 + x_2 > 0$.
6. $K(x_1, x_2) = x_1^2 + x_2^2$. F is the region in the first quadrant bounded by $x_1^2 - x_2^2 = 1$, $x_1^2 - x_2^2 = 2$, $x_1 x_2 = 1$, $x_1 x_2 = 2$. The inverse mapping is: $u_1 = x_1^2 - x_2^2$, $u_2 = 2x_1 x_2$.
7. Prove Lemma 14.8.
8. Complete the proof of Lemma 14.10.
9. Evaluate the integral
$$\int_F x_3\, dV_3(x)$$
by changing to spherical coordinates: $x_1 = \rho \cos\varphi \sin\theta$, $x_2 = \rho \sin\varphi \sin\theta$, $x_3 = \rho\cos\theta$, where F is the region determined by the inequalities $0 \leqslant x_1^2 + x_2^2 \leqslant x_3^2$, $0 \leqslant x_1^2 + x_2^2 + x_3^2 \leqslant 1$, $x_3 \geqslant 0$.

10. Write a proof of the Fundamental Lemma of Differentiation for vector functions (Theorem 14.5, Part (a)).

11. Show that the product of simple transformations is simple and that the inverse of a simple transformation is simple (Lemma 14.6).

12. Let $g = (g^1, g^2, \ldots, g^m)$ where $g^j = x_j$ for $j \neq i; j = 1, 2, \ldots, m$, and $g^i = \varphi(x)$. Show that $\nabla g = \varphi_{,i}$ (see Lemma 14.7).

13. If f is of class C^1 on a closed bounded region $\bar{G}$ in $\mathbb{R}^m$, show that det ∇f is uniformly continuous on $\bar{G}$.

14.4. The Lagrange Multiplier Rule

Let D be a region in $\mathbb{R}^m$ and suppose that $f: D \to \mathbb{R}^1$ is a C^1 function. At any local maximum or minimum of $f(x) = f(x_1, \ldots, x_m)$, we know that $f_{,i} = 0$, $i = 1, 2, \ldots, m$. In many applications we wish to find the local maxima and minima of such a function f subject to certain constraints. These constraints are usually given by a set of equations such as

$$\varphi^1(x_1, x_2, \ldots, x_m) = 0, \qquad \varphi^2(x_1, x_2, \ldots, x_m) = 0, \ldots,$$
$$\varphi^k(x_1, x_2, \ldots, x_m) = 0. \tag{14.37}$$

Equations (14.37) are called **side conditions**. Throughout we shall suppose that k is less than m. Otherwise, if there were say m side conditions, Equations (14.37) when solved simultaneously might yield a unique solution $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_m)$. Then this value when inserted in f would give a solution to the problem without further calculation. We reject the case $k > m$ since there may be no solution to the system given by (14.37). We shall suppose that the functions $\varphi^i: D \to \mathbb{R}^1$, $i = 1, 2, \ldots, k$, are C^1 functions, and furthermore that the $k \times m$ matrix

$$\begin{pmatrix} \varphi^1_{,1} & \varphi^1_{,2} \cdots \varphi^1_{,m} \\ \vdots & \vdots \\ \varphi^k_{,1} & \varphi^k_{,2} \cdots \varphi^k_{,m} \end{pmatrix}$$

is of rank k. That is, we suppose that at least one of the $k \times k$ minors of the above matrix has determinant different from zero in D. Without loss of generality, we assume that the square matrix consisting of the first k columns has nonvanishing determinant in D. This may always be achieved by relabeling the variables. Then according to the Implicit function theorem, in the neighborhood of any point $\bar{x} \in D$ we may solve for $x_1, x_2, \ldots, x_k$ in terms of $x_{k+1}, \ldots, x_m$. That is, there are functions $g^1, \ldots, g^k$ of class C^1 such that Equations (14.37) can be written

$$x_1 = g^1(x_{k+1}, \ldots, x_m), \quad x_2 = g^2(x_{k+1}, \ldots, x_m), \quad \ldots, \quad x_k = g^k(x_{k+1}, \ldots, x_m). \tag{14.38}$$

A customary way of finding a local maximum or minimum of f subject to the side Conditions (14.38) consists of the following procedure. First, solve the system given by (14.37) for $x_1, \ldots, x_k$ and obtain Equations (14.38). We assume that this is valid throughout D. Second, insert the functions in (14.38) in f, obtaining a function of the variables $x_{k+1}, \ldots, x_m$ given by

$$H(x_{k+1}, \ldots, x_m) \equiv f[g^1(x_{k+1}, \ldots, x_m), g^2(x_{k+1}, \ldots, x_m), \ldots, g^k(x_{k+1}, \ldots, x_m), x_{k+1}, \ldots, x_m].$$

Finally, find the local maxima and minima of H as a function of $x_{k+1}, \ldots, x_m$ in the ordinary way. That is, compute

$$H_{,i} = \sum_{j=1}^{k} f_{,j} g^j_{,i} + f_{,i}, \qquad i = k+1, \ldots, m, \tag{14.39}$$

and then find the solutions of the system of $m - k$ equations

$$H_{,i} = 0. \tag{14.40}$$

The values of $x_{k+1}, \ldots, x_m$ obtained in this way are inserted in (14.38) to yield values for $x_1, \ldots, x_k$. In this way we obtain the critical points of f which also satisfy (14.37). Various second derivative tests may then be used to decide whether the critical points are local maxima, local minima, or neither.

The method of **Lagrange multipliers** employs a simpler technique for achieving the same purpose. The method is especially useful when it is difficult or not possible to solve the system given by (14.37) in order to obtain the functions $g^1, \ldots, g^k$ given by (14.38).

The Lagrange multiplier rule is frequently explained but seldom proved. In Theorem 14.7 below we establish the validity of this rule which we now describe. We introduce k new variables (or parameters), denoted by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, and we form the function of $m + k$ variables

$$F(x, \lambda) = F(x_1, \ldots, x_m, \lambda_1, \ldots, \lambda_k) \equiv f(x) + \sum_{j=1}^{k} \lambda_j \varphi^j(x).$$

For this function F we compute the critical points when x is in D and λ in $\mathbb{R}^k$ *without* side conditions. That is, we find solutions to the $m + k$ equations formed by all the first derivatives of $F(x, \lambda)$:

$$F_{,i} = 0, \qquad i = 1, 2, \ldots, m,$$
$$F_{,j} = 0, \qquad j = 1, 2, \ldots, k. \tag{14.41}$$

We shall show that the critical points given by solutions of (14.40) are among the solutions of the system given by (14.41).

Suppose that f takes on its minimum at x^0, a point in the set D_0 consisting of all points x in D where the side conditions (14.37) hold. Suppose there is a function $g = (g^1, g^2, \ldots, g^m)$ from $I = \{t\colon -t_0 < t < t_0\}$ into $\mathbb{R}^m$ which is of class C^1 and has the properties

$$g(0) = x^0 \quad \text{and} \quad \phi^j[g(t)] = 0 \quad \text{for} \quad j = 1, 2, \ldots, k;\ t \in I. \tag{14.42}$$

Then the function $\Phi: I \to \mathbb{R}^m$ defined by

$$\Phi(t) = f[g(t)] \tag{14.43}$$

takes on its minimum at $t = 0$. Differentiating (14.42) and (14.43) with respect to t and setting $t = 0$, we get

$$\sum_{i=1}^{m} \phi^j_{,i}(x^0)\frac{dg^i(0)}{dt} = 0 \quad \text{and} \quad \sum_{i=1}^{m} f_{,i}(x_0)\frac{dg^i(0)}{dt} = 0. \tag{14.44}$$

Now let $h = (h_1, h_2, \ldots, h_m)$ be *any* vector[3] in V_m which is orthogonal to the k vectors $(\phi^j_{,1}(x^0), \phi^j_{,2}(x^0), \ldots, \phi^j_{,m}(x^0))$, $j = 1, 2, \ldots, k$. That is, suppose that

$$\sum_{i=1}^{m} \phi^j_{,i}(x^0)h_i = 0 \quad \text{or} \quad \nabla\phi^j(x^0)\cdot h = 0, \qquad j = 1, 2, \ldots, k.$$

From the Implicit function theorem, it follows that we may solve (14.37) for $x_1, \ldots, x_k$ in terms of $x_{k+1}, \ldots, x_m$, getting

$$x_i = \mu^i(x_{k+1}, \ldots, x_m), \qquad i = 1, 2, \ldots, k.$$

If we denote $x^0 = (x^0_1, \ldots, x^0_m)$ and define

$$g^i(t) = \begin{cases} \mu^i(x^0_{k+1} + th_{k+1}, \ldots, x^0_m + th_m), & i = 1, 2, \ldots, k, \\ x^0_i + th_i, & i = k+1, \ldots, m, \end{cases}$$

then $g = (g^1(t), \ldots, g^m(t))$ satisfies Conditions (14.42) and (14.44). We have thereby proved the following lemma.

Lemma 14.11. *Suppose that $f, \phi^1, \phi^2, \ldots, \phi^k$ are C^1 functions on an open set D in $\mathbb{R}^m$ containing a point x^0, that the vectors $\nabla\phi^1(x^0), \ldots, \nabla\phi^k(x^0)$ are linearly independent, and that f takes on its minimum among all points of D_0 at x^0, where D_0 is the subset of D on which the side conditions* (14.37) *hold. If h is any vector in V_m orthogonal to $\nabla\phi^1(x^0), \ldots, \nabla\phi^k(x^0)$, then*

$$\nabla f(x^0)\cdot h = 0.$$

The next lemma, concerning a simple fact about vectors in V_m, is needed in the proof of the Lagrange multiplier rule.

Lemma 14.12. *Let $b^1, b^2, \ldots, b^k$ be linearly independent vectors in the vector space V_m. Suppose that a is a vector in V_m with the property that a is orthogonal to any vector h which is orthogonal to all the b^i. Then there are numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that*

$$a = \sum_{i=1}^{k} \lambda_i b^i.$$

That is, a is in the subspace spanned by $b^1, b^2, \ldots, b^k$.

[3] In this argument we assume the reader is familiar with the customary m-dimensional vector space, denoted V_m. See Appendix 4.

PROOF. Let B be the subspace of V_m spanned by $b^1, b^2, \ldots, b^k$. Then there are vectors $c^{k+1}, c^{k+2}, \ldots, c^m$, such that the set $b^1, \ldots, b^k, c^{k+1}, \ldots, c^m$ form a linearly independent set (basis) of vectors in V_m. Let h be any vector orthogonal to all the b^i; then h will have components $h_1, \ldots, h_m$ in terms of the above basis with $h_1 = h_2 = \cdots = h_k = 0$. The vector a with components $(a_1, \ldots, a_m)$ and with the property $a \cdot h = 0$ for *all* such h must have $a_{i+1} = a_{i+2} = \cdots = a_m = 0$. Therefore, $a = \sum_{i=1}^k a_i b^i$. We set $a_i = \lambda_i$ to obtain the result. □

Theorem 14.7 (Lagrange multiplier rule). *Suppose that $f, \phi^1, \phi^2, \ldots, \phi^k$ and x^0 satisfy the hypotheses of Lemma* 14.11. *Define*

$$F(x, \lambda) = f(x) - \sum_{i=1}^{k} \lambda_i \phi^i(x).$$

Then there are numbers $\lambda_1^0, \lambda_2^0, \ldots, \lambda_k^0$ such that

$$F_{x_i}(x^0, \lambda^0) = 0, \qquad i = 1, 2, \ldots, m,$$

and

$$F_{\lambda_j}(x^0, \lambda^0) = 0, \qquad j = 1, 2, \ldots, k. \tag{14.45}$$

PROOF. The Equations (14.45) are

$$\nabla f(x_0) = \sum_{l=1}^{k} \lambda_l^0 \nabla \phi^l(x^0) \qquad \text{and} \qquad \phi^j(x^0) = 0, \qquad j = 1, 2, \ldots, k.$$

We set $a = \nabla f(x^0)$ and $b^j = \nabla \phi^j(x^0)$. Then Lemma 14.11 and 14.12 combine to yield the result. □

Remark. This theorem shows that the minimum (or maximum) of f subject to the side conditions $\phi^1 = \phi^2 = \cdots = \phi^k = 0$ is among the minima (or maxima) of the function F without any constraints.

EXAMPLE. Find the maximum of the function $x_1 + 3x_2 - 2x_3$ on the sphere $x_1^2 + x_2^2 + x_3^2 = 14$.

Solution. Let $F(x_1, x_2, x_3, \lambda) \equiv x_1 + 3x_2 - 2x_3 + \lambda(x_1^2 + x_2^2 + x_3^2 - 14)$. Then $F_{,1} = 1 + 2\lambda x_1$, $F_{,2} = 3 + 2\lambda x_2$, $F_{,3} = -2 + 2\lambda x_3$, $F_{,4} = x_1^2 + x_2^2 + x_3^2 - 14$. Setting $F_{,i} = 0$, $i = 1, \ldots, 4$, we obtain

$$x_1 = -\frac{1}{2\lambda}, \qquad x_2 = -\frac{3}{2\lambda}, \qquad x_3 = \frac{1}{\lambda}, \qquad 14 = \frac{14}{4\lambda^2}.$$

The solutions are $(x_1, x_2, x_3, \lambda) = (1, 3, -2, -\frac{1}{2})$ or $(-1, -3, 2, \frac{1}{2})$. The first solution gives the maximum value of 14. □

PROBLEMS

In each of Problems 1 through 10 find the solution by the Lagrange multiplier rule.

1. Find the minimum value of $x_1^2 + 3x_2^2 + 2x_3^2$ subject to the condition $2x_1 + 3x_2 + 4x_3 - 15 = 0$.

2. Find the minimum value of $2x_1^2 + x_2^2 + 2x_3^2$, subject to the condition $2x_1 + 3x_2 - 2x_3 - 13 = 0$.

3. Find the minimum value of $x_1^2 + x_2^2 + x_3^2$ subject to the conditions $2x_1 + 2x_2 + x_3 + 9 = 0$ and $2x_1 - x_2 - 2x_3 - 18 = 0$.

4. Find the minimum value of $4x_1^2 + 2x_2^2 + 3x_3^2$ subject to the conditions $x_1 + 2x_2 + 3x_3 - 9 = 0$ and $4x_1 - 2x_2 + x_3 + 19 = 0$.

5. Find the minimum value of $x_1^2 + x_2^2 + x_3^2 + x_4^2$ subject to the condition $2x_1 + x_2 - x_3 - 2x_4 - 5 = 0$.

6. Find the minimum value of $x_1^2 + x_2^2 + x_3^2 + x_4^2$ subject to the conditions $x_1 - x_2 + x_3 + x_4 - 4 = 0$ and $x_1 + x_2 - x_3 + x_4 + 6 = 0$.

7. Find the points on the curve $4x_1^2 + 4x_1x_2 + x_2^2 = 25$ which are nearest to the origin.

8. Find the points on the curve $7x_1^2 + 6x_1x_2 + 2x_2^2 = 25$ which are nearest to the origin.

9. Find the points on the curve $x_1^4 + y_1^4 + 3x_1y_1 = 2$ which are farthest from the origin.

10. Let $b_1, b_2, \ldots, b_k$ be positive numbers. Find the maximum value of $\sum_{i=1}^k b_i x_i$ subject to the side condition $\sum_{i=1}^k x_i^2 = 1$.

11. (a) Find the maximum of the function $x_1^2 \cdot x_2^2 \cdots x_n^2$ subject to the side condition $\sum_{i=1}^n x_i^2 = 1$.
    (b) If $\sum_{i=1}^n x_i^2 = 1$, show that $(x_1^2 x_2^2 \cdots x_n^2)^{1/n} \leqslant 1/n$.
    (c) If $a_1, a_2, \cdots, a_n$ are positive numbers, prove that

$$(a_1 \cdot a_2 \cdots a_n)^{1/n} \leqslant \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

[The geometric mean of n numbers is always less than or equal to the arithmetic mean.]