In the following, we write $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_{++} = (0, \infty)$. So $\mathbb{R}_+^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0\}$ and $\mathbb{R}_{++}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i > 0\}$.

1. (a) Find all stationary points of the cubic polynomial

$$f(x, y) = x^3 + y^3 - 3x - 12y + 20.$$ 

Indicate which are the local maximizers and local minimizers.

(b) A rectangular box, open at the top, has a volume of 32 cubic feet. Find the dimensions of the box so that the total surface area is minimized.

(c) Prove that for any $x \geq 0$, $y \geq 0$, we always have

$$x^2 + y^2 \leq e^{x+y} - 2.$$ 

(d) Let $A \in \mathbb{S}^n$, $b \in \mathbb{R}^n$, and $c \in \mathbb{R}$. Show that $\frac{1}{2}x^T A x + b^T x + c$ has a unique global minimizer iff $A \succ 0$. What is it?

(e) Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Show that $x^*$ is a global minimizer of $\|Ax - b\|_2^2$ iff $x^*$ is a solution to $A^T A x = A^T b$.

2. Compute the Hessians of the following functions and decide if they are convex, concave, or neither on their respective domains.

(a) $e, f, g, h : \mathbb{R}_{++}^2 \to \mathbb{R}$ defined by

$$e(x, y) = xy, \quad f(x, y) = \frac{1}{x+y}, \quad g(x, y) = \frac{x}{y}, \quad h(x, y) = x^\alpha y^{1-\alpha},$$

where $\alpha \in [0, 1]$.

(b) $\varphi : \mathbb{R} \times \mathbb{R}_{++} \to \mathbb{R}$ defined by $\varphi(x, y) = x^2/y$. (Hint: Write $\nabla^2 \varphi(x, y)$ as a rank-1 matrix).

(c) Approximate $e, f, g, \varphi$ by a 3-term Taylor series about the point $x = [\frac{3}{2}, \frac{1}{2}]$ and the value of its corresponding 3-term Taylor series approximation.

3. (a) Find the Hessian of the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(x) = \log(e^{x_1} + \cdots + e^{x_n})$. Show that for any $v \in \mathbb{R}^n$,

$$v^T \nabla^2 f(x)v = \frac{1}{(e^{x_1} + \cdots + e^{x_n})^2} \left[ \left( \sum_{i=1}^n x_i \right) \left( \sum_{i=1}^n v_i e^{x_i} \right) - \left( \sum_{i=1}^n v_i e^{x_i} \right)^2 \right].$$

Hence or otherwise, deduce that $f$ is a convex function.

(b) Find the Hessian of the function $g : \mathbb{R}_{++}^n \to \mathbb{R}$ defined by $g(x) = (x_1 \cdots x_n)^{1/n}$. Show that for any $v \in \mathbb{R}^n$,

$$v^T \nabla^2 g(x)v = -\frac{g(x)}{n^2} \left[ n \sum_{i=1}^n \frac{v_i^2}{x_i} - \left( \sum_{i=1}^n \frac{v_i}{x_i} \right)^2 \right].$$

Hence or otherwise, deduce that $g$ is a concave function.

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(c) Find the Hessian of the function $h : \mathbb{R}^n_+ \to \mathbb{R}$ defined by

$$h(x) = \frac{1}{1/x_1 + \cdots + 1/x_n}.$$

By emulating what we did in the previous two parts or otherwise, decide if $h$ is convex, concave, or neither on $\mathbb{R}^n_+$.

(d) Find the Hessian of the function $\varphi : \mathbb{R}^n_+ \to \mathbb{R}$ defined by $\varphi(x) = \log h(x)$. Decide if $\varphi$ is convex, concave, or neither on $\mathbb{R}^n_+$.

4. (a) Show that the negative log function $-\log : \mathbb{R}_+ \to \mathbb{R}$ is strictly convex, i.e.,

$$\log(tx + (1-t)y) > t \log x + (1-t) \log y$$  

for any $x, y \in \mathbb{R}_+$ and any $t \in (0, 1)$.

(b) Prove the generalized arithmetic-geometric mean inequality

$$a^t b^{1-t} \leq ta + (1-t)b$$

for any $a, b \in \mathbb{R}_+$ and $t \in [0, 1]$ (note that $t = 1/2$ gives us the usual arithmetic-geometric mean inequality). Deduce the Hölder inequality: for $p > 1$ and $1/p + 1/q = 1$,

$$\sum_{i=1}^n |x_i y_i| \leq \left( \sum_{i=1}^n |x_i|^p \right)^{1/p} \left( \sum_{i=1}^n |y_i|^q \right)^{1/q}$$

for any $x, y \in \mathbb{R}^n$ (note that $p = q = 2$ gives us the Cauchy–Schwartz inequality).

(c) Show that $(\sin \theta)^{\sin \theta} < (\cos \theta)^{\cos \theta}$ for all $\theta \in (0, \pi/4)$. 