STAT 280: OPTIMIZATION
SPRING 2022
PROBLEM SET 1

1. (a) Let $m, n \in \mathbb{N}$, i.e., positive integers. Find all local and global optimizers, if any, of the following functions on $\mathbb{R}$,

$$f(x) = \left(1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}\right) e^{-x}, \quad g(x) = x^m(1-x)^n, \quad h(x) = \sin^{2m} x \cdot \cos^{2n} x.$$

(b) Find the local and global optimizers, if any, of f on $\mathbb{R}$ and of g on $[-1, 1]$,

$$f(x) = x^{1/3}(1-x)^{2/3}, \quad g(x) = x \sin^{-1} x + \sqrt{1 - x^2}.$$

(c) Find the global minimum of the following functions on $\mathbb{R}$,

$$f(x) = -\frac{1}{1+|x|} - \frac{1}{1+|x-1|}, \quad g(x) = \begin{cases} e^{-\frac{1}{|x|}}\left(\sqrt{2} + \frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

2. The key to the problems below is to figure out the right x to choose so that you get a univariate optimization problem.

(a) Which number is larger, π^e or e^π?

(b) Show that for any $p \geq 1$ and $a, b \in [0, 1]$,

$$|a^p - b^p| \leq p|a - b|.$$

(c) Write down an expression for the area of a rectangle inscribed in the unit circle as a differentiable univariate function $f(x)$. Hence show that the largest inscribed rectangle in a circle must be a square.

(d) Light travels at different speeds in different media (e.g. air and water). Consider the following scenario depicted in Figure 1. Let v_1 be the speed of light in air and v_2 be the speed of light in water. Write down an expression for time required for light to travel from a point in air to a point in water. Show that the minimum is attained when

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}.$$

3. Let $f \in C^2([a, b])$, $f(a)f(b) < 0$, f' and f'' do not vanish and do not change signs on $[a, b]$.

(a) Show that $f(x) = 0$ has a unique solution $x_* \in (a, b)$.

(b) Prove if f' and f'' have opposite signs, then Newton method starting at $x_0 = a$ must converge to x_*.

(c) What if f' and f'' have the same sign, which starting point should we choose to guarantee convergence? Draw pictures to illustrate four different cases: (i) $f' > 0$, $f'' < 0$, (ii) $f' < 0$, $f'' > 0$, (iii) $f' > 0$, $f'' > 0$, (iv) $f' < 0$, $f'' < 0$.

(d) Let $M = \max\{|f''(x)| : x \in [a, b]\}$ and $m = \min\{|f'(x)| : x \in [a, b]\}$. Show that the errors $e_n = x_n - x_*$, $n = 0, 1, 2, \ldots$, satisfy

$$|e_{n+1}| \leq \frac{M}{2m}|e_n|^2.$$

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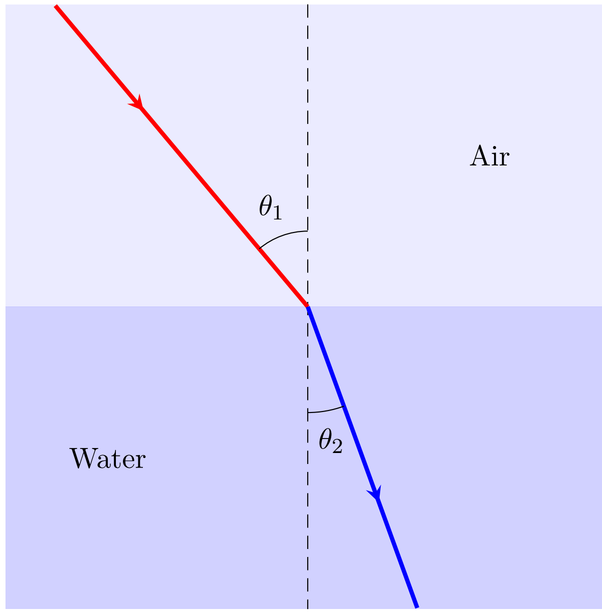

FIGURE 1. Light passing through different media.

4. (a) Let $g \in C^1([a,b])$. Suppose $g(x) \in [a,b]$ and $|g'(x)| < 1$ for all $x \in [a,b]$. Show that (i) $g(x) = x$ has a unique solution $x_* \in [a,b]$, (ii) the iteration $x_{n+1} = g(x_n)$ must converge for any $x_0 \in [a,b]$, and (iii) $\lim_{n\to\infty} x_n = x_*$.

(b) Let $f \in C^2([a,b])$ and x_* be a simple root of f in (a,b), i.e., $f'(x_*) \neq 0$. Apply (a) to show that there is a neighborhood of x_* where Newton method must converge if x_0 is chosen in this neighborhood.

(c) Suppose $g \in C([a,b])$. Show that if the iteration $x_{n+1} = g(x_n)$ converges to $x_* \in [a,b]$ for any $x_0 \in [a,b]$, then we must have $|g'(x_*)| \le 1$. Find an example where $g'(x_*) = 1$.

5. In all of the following you will need to justify your answers.

(a) State whether the following sequences converge sublinearly, linearly, superlinearly, or quadratically (give the fastest rate):

$$\left(\frac{1}{n^2}\right)_{n=1}^{\infty}, \quad \left(\frac{1}{2^{2^n}}\right)_{n=1}^{\infty}, \quad \left(\frac{1}{\sqrt{n}}\right)_{n=1}^{\infty}, \quad (e^{-n})_{n=1}^{\infty}, \quad \left(\frac{1}{n^n}\right)_{n=1}^{\infty}.$$

(b) Let $x_{n+1} = x_n + x_{n-1}$ with $x_0 = 0$ and $x_1 = 1$. Let $y_n := x_n/x_{n-1}$. Is $(y_n)_{n=1}^{\infty}$ convergent? If so, is the convergence sublinear, linear, superlinear, or quadratic?

(c) Design an algorithm for computing reciprocal of a positive real number $a > 0$ that requires only addition and multiplication. For what values of x_0 do the algorithm converges? Apply your algorithm to find the decimal expansion of $1/12$ to 10 decimal digits of accuracy starting from $x_0 = 0.1$ and $x_0 = 1$. Discuss your results.

(d) Find the number of roots of $2x = 3\sin x$ in $\mathbb{R}$ and determine the largest root to three decimal accuracy.