1. (a) Let \(m, n \in \mathbb{N}\). Find all local and global extrema, if any, of the following functions on \(\mathbb{R}\),
\[
f(x) = \left(1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}\right) e^{-x}, \quad g(x) = x^n(1-x)^n, \quad h(x) = \sin^{2m} x \cdot \cos^{2n} x.
\]
(b) Find the local and global extrema, if any, of \(f\) on \(\mathbb{R}\) and of \(g\) on \([-1,1]\),
\[
f(x) = x^{1/3}(1-x)^{2/3}, \quad g(x) = x^{-1} + \sqrt{1-x^2}.
\]
(c) Find the global minimum of the following functions on \(\mathbb{R}\),
\[
f(x) = -\frac{1}{1+|x|}, \quad g(x) = \begin{cases} e^{-\frac{1}{|x|}} \left(\sqrt{2} + \frac{1}{2}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}
\]

2. Let \(f \in C^2([a,b])\).
   (a) Suppose there exists a constant \(\alpha > 0\) such that \(f''(x) = \alpha f(x)\). Show that
   \[
   |f(x)| \leq \max\{|f(a)|, |f(b)|\}
   \]
   for all \(x \in [a,b]\).
   (b) Suppose \(f''(x) = e^x f(x)\). Show that \(f\) cannot have a positive local maximum or a negative
   local minimum in \((a,b)\). Deduce that if \(f(a) = f(b) = 0\), then \(f\) vanishes identically.
   (c) Suppose \(A := \sup_{x \in \mathbb{R}} |f(x)|, B := \sup_{x \in \mathbb{R}} |f'(x)|,\) and \(C := \sup_{x \in \mathbb{R}} |f''(x)|\) are finite. Prove that
   \[
   B \leq 2 \sqrt{AC}.
   \]
   
   (Hint: Consider \(f(x+2h)\) for an appropriate \(h\) and apply Taylor’s theorem.)

3. The key to the problems below is to figure out the right \(x\) to choose so that you get a univariate
   optimization problem.
   (a) Which number is larger, \(\pi e\) or \(e \pi\)?
   (b) Write down an expression for the area of a rectangle inscribed in the unit circle as a
   differentiable univariate function \(f(x)\). Hence show that the largest inscribed rectangle in
   a circle must be a square.
   (c) Light travels at different speeds in different media (e.g. air and water). Consider the
   following scenario depicted in Figure 1. Let \(v_1\) be the speed of light in air and \(v_2\) be the
   speed of light in water. Write down an expression for time required for light to travel from
   a point in airt to a point in water. Show that the minimum is attained when
   \[
   \frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}.
   \]

4. Let \(f \in C^2([a,b]), f(a)f(b) < 0,\) and \(f'\) and \(f''\) do not change signs on \([a,b]\).
   (a) Show that \(f(x) = 0\) has a unique solution \(x_* \in (a,b)\).
   (b) Prove if \(f'\) and \(f''\) have opposite signs, then Newton method starting at \(x_0 = a\) must
   converge to \(x_*\).
(c) What if $f'$ and $f''$ have the same sign, which starting point should we choose to guarantee convergence? Draw pictures to illustrate four different cases: (i) $f' > 0$, $f'' < 0$, (ii) $f' < 0$, $f'' > 0$, (iii) $f' > 0$, $f'' > 0$, (iv) $f' < 0$, $f'' < 0$.

(d) Let $M = \max \{ |f''(x)| : x \in [a,b] \}$ and $m = \min \{ |f'(x)| : x \in [a,b] \}$. Show that the errors $e_n = x_n - x*$, $n = 0, 1, 2, \ldots$, satisfy

$$|e_{n+1}| \leq \frac{M}{2m} |e_n|^2.$$

5. In all of the following you will need to justify your answers.

(a) State whether the following sequences converge sublinearly, linearly, superlinearly, or quadratically (give the fastest rate):

$$\left( \frac{1}{n^2} \right)_{n=1}^{\infty}, \left( \frac{1}{2^n} \right)_{n=1}^{\infty}, \left( \frac{1}{\sqrt{n}} \right)_{n=1}^{\infty}, (e^{-n})_{n=1}^{\infty}, \left( \frac{1}{n^n} \right)_{n=1}^{\infty}.$$

(b) Design an algorithm for computing reciprocal of a positive real number $a > 0$ that requires only addition and multiplication. For what values of $x_0$ does the algorithm converge? Apply your algorithm to find the decimal expansion of $1/12$ to 10 decimal digits of accuracy starting from $x_0 = 0.1$ and $x_0 = 1$. Discuss your results.