1. (a) Let \( m, n \in \mathbb{N} \). Find all local and global extrema, if any, of the following functions on \( \mathbb{R} \),
\[
f(x) = \left(1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}\right) e^{-x}, \quad g(x) = x^n (1 - x)^m, \quad h(x) = \sin^{2m} x \cdot \cos^{2n} x.
\]
(b) Find the local and global extrema, if any, of \( f \) on \( \mathbb{R} \) and of \( g \) on \([-1, 1]\),
\[
f(x) = x^{1/3} (1 - x)^{2/3}, \quad g(x) = x \sin^{-1} x + \sqrt{1 - x^2}.
\]
(c) Find the global minimum of the following functions on \( \mathbb{R} \),
\[
f(x) = -\frac{1}{1 + |x|} - \frac{1}{1 + |x - 1|}, \quad g(x) = \begin{cases} e^{-\frac{1}{x^2}} (\sqrt{2} + \frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}
\]

2. Let \( f \in C^2([a, b]) \).
   (a) Suppose there exists a constant \( \alpha > 0 \) such that \( f''(x) = \alpha f(x) \). Show that
   \[
   |f(x)| \leq \max\{|f(a)|, |f(b)|\}
   \]
   for all \( x \in [a, b] \).
   (b) Suppose \( f''(x) = e^x f(x) \). Show that \( f \) cannot have a positive local maximum or a negative
   local minimum in \((a, b)\). Deduce that if \( f(a) = f(b) = 0 \), then \( f \) vanishes identically.
   (c) Suppose \( A := \sup_{x \in \mathbb{R}} |f(x)|, B := \sup_{x \in \mathbb{R}} |f'(x)|, \) and \( C := \sup_{x \in \mathbb{R}} |f''(x)| \) are finite. Prove that
   \[
   B \leq 2\sqrt{AC}.
   \]
   (Hint: Consider \( f(x + 2h) \) for an appropriate \( h \) and apply Taylor’s theorem.)

3. The key to the problems below is to figure out the right \( x \) to choose so that you get a univariate
   optimization problem.
   (a) Which number is larger, \( \pi^e \) or \( e^\pi \)?
   (b) Write down an expression for the area of a rectangle inscribed in the unit circle as a
differentiable univariate function \( f(x) \). Hence show that the largest inscribed rectangle in
   a circle must be a square.
   (c) Light travels at different speeds in different media (e.g. air and water). Consider the
   following scenario depicted in Figure 1. Let \( v_1 \) be the speed of light in medium 1 and \( v_2 \)
   be the speed of light in medium 2. Write down an expression for time required for light
   to travel from a point in medium 1 to a point in medium 2. Show that the minimum is
   attained when
   \[
   \frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}.
   \]

4. Let \( f \in C^2([a, b]) \), \( f(a) f(b) < 0 \), and \( f' \) and \( f'' \) do not change signs on \([a, b]\).
   (a) Show that \( f(x) = 0 \) has a unique solution \( x_* \in (a, b) \).
   (b) Prove if \( f' \) and \( f'' \) have opposite signs, then Newton method starting at \( x_0 = a \) must
   converge to \( x_* \).

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(c) What if $f'$ and $f''$ have the same sign, which starting point should we choose to guarantee convergence? Draw pictures to illustrate four different cases: (i) $f' > 0$, $f'' < 0$, (ii) $f' < 0$, $f'' > 0$, (iii) $f' > 0$, $f'' > 0$, (iv) $f' < 0$, $f'' < 0$.

(d) Let $M = \max\{|f''(x)| : x \in [a, b]\}$ and $m = \min\{|f'(x)| : x \in [a, b]\}$. Show that the errors $e_n = x_n - x^*$, $n = 0, 1, 2, \ldots$, satisfy

$$|e_{n+1}| \leq \frac{M}{2m}|e_n|^2.$$

5. In all of the following you will need to justify your answers.

(a) State whether the following sequences converge linearly, superlinearly, or quadratically (give the fastest rate):

$$\left(\frac{1}{n^2}\right)_{n=1}^\infty, \quad \left(\frac{1}{2^n}\right)_{n=1}^\infty, \quad \left(\frac{1}{\sqrt{n}}\right)_{n=1}^\infty, \quad (e^{-n})_{n=1}^\infty, \quad \left(\frac{1}{n^n}\right)_{n=1}^\infty.$$

(b) Give an example of a convergent sequence whose convergence rate is slower than linear.

(c) Design an algorithm for computing reciprocal of a positive real number $a > 0$ that requires only addition and multiplication. For what values of $x_0$ do the algorithm converges? Apply your algorithm to find the decimal expansion of $1/12$ to 10 decimal digits of accuracy starting from $x_0 = 0.1$ and $x_0 = 1$. Discuss your results.