1. (a) Let $m, n \in \mathbb{N}$. Find all local and global extrema, if any, of the following functions on $\mathbb{R}$,
$$f(x) = \left(1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}\right) e^{-x}, \quad g(x) = x^m(1-x)^n, \quad h(x) = \sin^{2m} x \cdot \cos^{2n} x.$$ 
(b) Find the global extrema, if any, of $f$ on $\mathbb{R}$ and of $g$ on $[-1, 1],
$$f(x) = x^{1/3}(1-x)^{2/3}, \quad g(x) = x \sin^{-1} x + \sqrt{1-x^2}.$$
(c) Find the global minimum of the following functions on $\mathbb{R},
$$f(x) = -\frac{1}{1+|x|} - \frac{1}{1+|x-1|}, \quad g(x) = \begin{cases} e^{-\frac{|x|}{\sqrt{2}}} (\sqrt{2} + \frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

2. Let $f \in C^2([a,b]).$
(a) Suppose there exists a constant $\alpha > 0$ such that $f''(x) = \alpha f(x).$ Show that
$$|f(x)| \leq \max\{|f(a)|, |f(b)|\}$$
for all $x \in [a, b].$
(b) Suppose $f''(x) = e^x f(x).$ Show that $f$ cannot have a positive local maximum or a negative local minimum in $(a,b).$ Deduce that if $f(a) = f(b) = 0,$ then $f$ vanishes identically.
(c) Let $A := \sup_{x \in [a,b]} |f(x)|, \quad B := \sup_{x \in [a,b]} |f'(x)|, \quad \text{and} \quad C := \sup_{x \in [a,b]} |f''(x)|.$ Prove that
$$B \leq 2\sqrt{AC}.$$ 
(Hint: Consider $f(x + 2h)$ for an appropriate $h$ and apply Taylor’s theorem.)

3. The key to the problems below is to figure out the right $x$ to choose so that you get a univariate optimization problem.
(a) Which number is larger, $\pi e$ or $e^\pi$?
(b) Write down an expression for the area of a rectangle inscribed in the unit circle as a differentiable univariate function $f(x).$ Hence show that the largest inscribed rectangle in a circle must be a square.
(c) Light travels at different speeds in different media (e.g. air and water). Consider the following scenario depicted in Figure 1. Let $v_1$ be the speed of light in medium 1 and $v_2$ be the speed of light in medium 2. Write down an expression for time required for light to travel from a point in medium 1 to a point in medium 2. Show that the minimum is attained when
$$\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}.$$ 

4. Let $f \in C^2([a,b]), f(a)f(b) < 0,$ and $f'$ and $f''$ do not change signs on $[a,b].$
(a) Show that $f(x) = 0$ has a unique solution $x_\ast \in (a, b)$.
(b) Prove if $f'$ and $f''$ have opposite signs, then Newton method starting at $x_0 = a$ must converge to $x_\ast.$
(c) What if $f'$ and $f''$ have the same sign, which starting point should we choose to guarantee convergence? Draw pictures to illustrate four different cases: (i) $f' > 0$, $f'' < 0$, (ii) $f' < 0$, $f'' > 0$, (iii) $f' > 0$, $f'' > 0$, (iv) $f' < 0$, $f'' < 0$.

(d) Let $M = \max\{|f''(x)| : x \in [a, b]\}$ and $m = \min\{|f'(x)| : x \in [a, b]\}$. Show that the errors $e_n = x_n - x_*$, $n = 0, 1, 2, \ldots$, satisfy
\[ |e_{n+1}| \leq \frac{M}{2m} |e_n|^2. \]

5. In all of the following you will need to justify your answers.

(a) State whether the following sequences converge linearly, superlinearly, or quadratically (give the fastest rate):
\[ \left( \frac{1}{n^2} \right)_{n=1}^\infty, \quad \left( \frac{1}{2^n} \right)_{n=1}^\infty, \quad \left( \frac{1}{\sqrt{n}} \right)_{n=1}^\infty, \quad (e^{-n})_{n=1}^\infty, \quad \left( \frac{1}{n^n} \right)_{n=1}^\infty. \]

(b) Give an example of a convergent sequence whose convergence rate is slower than linear.

(c) Design an algorithm for computing reciprocal of a positive real number $a > 0$ that requires only addition and multiplication. For what values of $x_0$ do the algorithm converge? Apply your algorithm to find the decimal expansion of $1/12$ to 10 decimal digits of accuracy starting from $x_0 = 0.1$ and $x_0 = 1$. Discuss your results.

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**Figure 1.** Light passing through different media.