For a real-valued function of two real variables, \( u : \Omega _\mathbb{R} \to \mathbb{R} \), we say that \( u \) is \textit{twice continuously differentiable} if all second-order partial derivatives \( u_{xx}, u_{yy}, u_{xy}, u_{yx} \) exist and are continuous on \( \Omega _\mathbb{R} \). The set of all twice continuously differentiable functions on \( \Omega _\mathbb{R} \) is denoted \( C^2(\Omega _\mathbb{R}) \).

1. We mentioned Tauberian theorems in class. Here is an example of an easy one (easy relative to other Tauberian theorems). Let \( \sum _{n=0}^{\infty} a_n z^n \) be a power series with radius of convergence 1 and suppose
\[
\lim _{n \to \infty} n a_n = 0.
\]
(a) Show that
\[
\lim _{m \to \infty} \sum _{n=0}^{m} n |a_n| = 0.
\]
(\textit{Hint: Problem 4(a), Problem Set 3, Math 104, Spring 2009.})
(b) Define a function \( f \) by
\[
f(z) = \sum _{n=0}^{\infty} a_n z^n \quad \text{for all } |z| < 1.
\]
Let \( x \) be a real variable and suppose the following left limit exists
\[
\lim _{x \to 1^-} f(x) = A.
\]
Show that the series \( \sum _{n=0}^{\infty} a_n \) converges to \( A \).

2. Recall that \( \mathbb{C} \) is both a real vector space of dimension 2 and a complex vector space of dimension 1. A function \( \varphi : \mathbb{C} \to \mathbb{C} \) is called \textit{\( \mathbb{R} \)-linear} if \( \varphi \) is a linear transformation of real vector spaces, ie.
\[
\varphi(\lambda_1 z_1 + \lambda_2 z_2) = \lambda_1 \varphi(z_1) + \lambda_2 \varphi(z_2) \quad \text{for all } \lambda_1, \lambda_2 \in \mathbb{R} \text{ and } z_1, z_2 \in \mathbb{C}.
\]
It is called \textit{\( \mathbb{C} \)-linear} if \( \varphi \) is a linear transformation of complex vector spaces, ie.
\[
\varphi(\lambda_1 z_1 + \lambda_2 z_2) = \lambda_1 \varphi(z_1) + \lambda_2 \varphi(z_2) \quad \text{for all } \lambda_1, \lambda_2 \in \mathbb{C} \text{ and } z_1, z_2 \in \mathbb{C}.
\]
(a) Prove that if \( \varphi \) is \( \mathbb{C} \)-linear, then it is \( \mathbb{R} \)-linear. Give an example to show that the converse is false.
(b) Let \( \varphi : \mathbb{C} \to \mathbb{C} \). Prove that the following statements are equivalent.
   (i) \( \varphi \) is \( \mathbb{R} \)-linear.
   (ii) \( \varphi \) satisfies
   \[
   \varphi(z) = \varphi(1)x + \varphi(i)y
   \]
   for all \( z = x + iy \in \mathbb{C} \).
   (iii) \( \varphi \) satisfies
   \[
   \varphi(z) = \left[ \frac{\varphi(1) - i \varphi(i)}{2} \right] z + \left[ \frac{\varphi(1) + i \varphi(i)}{2} \right] \bar{z}
   \]
   for all \( z = x + iy \in \mathbb{C} \).
\(^{(iv)}\) \(\varphi\) is given by
\[ \varphi(x + iy) = (ax + by) + i(cx + dy) \]
for some \(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2 \times 2}. \)

(c) Let \(\varphi : \mathbb{C} \to \mathbb{C}\). Prove that the following statements are equivalent.

(i) \(\varphi\) is \(\mathbb{C}\)-linear.
(ii) \(\varphi\) is \(\mathbb{R}\)-linear and \(\varphi(i) = i\varphi(1)\).
(iii) \(\varphi\) satisfies
\[ \varphi(z) = \varphi(1)z \]
for all \(z \in \mathbb{C}\).
(iv) \(\varphi\) is given by
\[ \varphi(x + iy) = (ax - cy) + i(cx + ay) \]
for some \(\begin{bmatrix} a & c \\ -c & a \end{bmatrix} \in \mathbb{R}^{2 \times 2}. \)

3. Let \(\Omega \subseteq \mathbb{C}\) be a region and let \(f : \Omega \to \mathbb{C}\). We will call \(f\) \textit{complex differentiable} at \(z \in \Omega\) if it is differentiable as defined in the lectures, ie. the limit
\[ \lim_{h \to 0} \frac{f(z + h) - f(z)}{h} \]  
exists. We will call \(f\) \textit{real differentiable} at \(z \in \Omega\) if there exists a \(\mathbb{R}\)-linear function \(\varphi : \mathbb{C} \to \mathbb{C}\) such that
\[ \lim_{h \to 0} \frac{f(z + h) - f(z) - \varphi(h)}{h} = 0. \]  

(a) Prove that if \(f\) is complex differentiable at \(z \in \Omega\), then \(f\) is real differentiable at \(z\).
(b) Give an example to show that the converse of (a) is false.
(c) Let \(f\) be real differentiable at \(z \in \Omega\). If the \(\mathbb{R}\)-linear function \(\varphi : \mathbb{C} \to \mathbb{C}\) in (3.2) is also \(\mathbb{C}\)-linear, prove that \(f\) is complex differentiable at \(z\). In this case, how is \(\varphi\) related to the limit in (3.1)?
(d) Let \(f\) be real differentiable at \(z \in \Omega\). Show that if the limit
\[ \lim_{h \to 0} \left| \frac{f(z + h) - f(z)}{h} \right| \]  
exists\(^1\), then either \(f\) or \(\bar{f}\) must be complex differentiable at \(z\). Give an example to show that \(f\) is not necessarily complex differentiable at \(z\). Here the function \(\bar{f} : \Omega \to \mathbb{C}\) is defined by \(f(z) = \bar{f}(\bar{z})\) for all \(z \in \Omega\).

4. (a) Show that the function \(f : \mathbb{C} \to \mathbb{C}\) defined by
\[ f(z) = \sqrt{|z^2 - \bar{z}^2|} \]
satisfies the Cauchy-Riemann equation at \(z = 0\) but is not differentiable at \(z = 0\).
(b) Let \(\Omega \subseteq \mathbb{C}\) be a region such that the function
\[ f(x + iy) = |x^2 - y^2| + 2i|xy| \]
is analytic on \(\Omega\) but is not analytic on any larger region \(\Omega'\) containing \(\Omega\). Find all possible \(\Omega\) with this property.
(c) Find constants \(a, b, c \in \mathbb{R}\) such that the functions \(f, g : \mathbb{C} \to \mathbb{C}\) defined by
\[ f(x + iy) = x + ay + i(bx + cy), \]
\[ g(x + iy) = \cos x(\cosh y + a \sinh y) + i \sin x(\cosh y + b \sinh y) \]
are analytic on \(\mathbb{C}\).

\(^1\)Note the difference between (3.1) and (3.3).
5. Let $\Omega \subseteq \mathbb{C}$ be a region. Let $f : \Omega \rightarrow \mathbb{C}$ be analytic and $u(x, y) = \text{Re } f(x + iy)$, $v(x, y) = \text{Im } f(x + iy)$.

(a) Suppose $u, v \in C^2(\Omega_R)$. Show that $u$ and $v$ are harmonic functions, i.e. solutions of the Laplace equation
$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0,$$
on $\Omega_R$.

(b) Let $a \in \mathbb{R}$. Suppose $f$ is analytic on $D(0, 1)$. Which of the following can occur as the real or imaginary part of $f$?

$$x^2 - axy + y^2, \quad x^3 - x^2 + y^3, \quad x^2 + y^2 - 5x, \quad \frac{x^2 - y^2}{(x^2 + y^2)^2}.$$

6. We may rewrite any complex function $f$ of two real variables $x$ and $y$ as a function of $z$ and $\overline{z}$ via

$$x = \frac{z + \overline{z}}{2}, \quad y = \frac{z - \overline{z}}{2i}.$$

(a) Considering $z$ and $\overline{z}$ as independent variables, show that

$$\frac{\partial f}{\partial z} = \frac{1}{2} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right) \quad \text{and} \quad \frac{\partial f}{\partial \overline{z}} = \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right).$$

(b) Show that the Cauchy-Riemann equation may be expressed as

$$\frac{\partial f}{\partial \overline{z}} = 0.$$

This may be interpreted as saying that complex differentiable functions must be independent\(^2\) of $\overline{z}$ and depend only on $z$.

(c) Which of the following complex functions of two real variables can be expressed in terms of a polynomial in $z = x + iy$?

$$f_1(x, y) = x^2 - y^2 - ixy, \quad f_2(x, y) = x^2 + y^2 - 2ixy.$$

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\(^2\)In fact you may also view this as a reason why there isn’t a ‘quaternion analysis’ similar to complex analysis. For a quaternion $q = x + yi + zj + wk$, its quaternionic conjugate $\overline{q} = x - yi - zj - wk$ can always be expressed in terms of $q$:

$$\overline{q} = \frac{1}{2} (q + iqj + jqk),$$

and so we don’t have functions dependent on $q$ but not on $\overline{q}$. 