MATH 185: COMPLEX ANALYSIS
FALL 2009/10
PROBLEM SET 3

For a real-valued function of two real variables, $u : \Omega_{\mathbb{R}} \to \mathbb{R}$, we say that u is *twice continuously differentiable* if all second-order partial derivatives $u_{xx}, u_{yy}, u_{xy}, u_{yx}$ exist and are continuous on $\Omega_{\mathbb{R}}$. The set of all twice continuously differentiable functions on $\Omega_{\mathbb{R}}$ is denoted $C^2(\Omega_{\mathbb{R}})$.

1. We mentioned Tauberian theorems in class. Here is an example of an easy one (easy relative to other Tauberian theorems). Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence 1 and suppose

$$\lim_{n\to\infty} n a_n = 0.$$

(a) Show that

$$\lim_{m\to\infty} \frac{\sum_{n=0}^{m} n|a_n|}{m} = 0.$$

(*Hint:* Problem **4**(a), Problem Set **3**, Math **104**, Spring 2009.)

(b) Define a function f by

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{for all } |z| < 1.$$

Let x be a real variable and suppose the following left limit exists

$$\lim_{x\to 1^-} f(x) = A.$$

Show that the series $\sum_{n=0}^{\infty} a_n$ converges to A.

2. Recall that $\mathbb{C}$ is both a real vector space of dimension 2 and a complex vector space of dimension 1. A function $\varphi : \mathbb{C} \to \mathbb{C}$ is called $\mathbb{R}$-*linear* if φ is a linear transformation of real vector spaces, ie.

$$\varphi(\lambda_1 z_1 + \lambda_2 z_2) = \lambda_1 \varphi(z_1) + \lambda_2 \varphi(z_2) \qquad \text{for all } \lambda_1, \lambda_2 \in \mathbb{R} \text{ and } z_1, z_2 \in \mathbb{C}.$$

It is called $\mathbb{C}$-*linear* if φ is a linear transformation of complex vector spaces, ie.

$$\varphi(\lambda_1 z_1 + \lambda_2 z_2) = \lambda_1 \varphi(z_1) + \lambda_2 \varphi(z_2) \qquad \text{for all } \lambda_1, \lambda_2 \in \mathbb{C} \text{ and } z_1, z_2 \in \mathbb{C}.$$

(a) Prove that if φ is $\mathbb{C}$-linear, then it is $\mathbb{R}$-linear. Give an example to show that the converse is false.

(b) Let $\varphi : \mathbb{C} \to \mathbb{C}$. Prove that the following statements are equivalent.

(i) φ is $\mathbb{R}$-linear.

(ii) φ satisfies

$$\varphi(z) = \varphi(1)x + \varphi(i)y$$

for all $z = x + iy \in \mathbb{C}$.

(iii) φ satisfies

$$\varphi(z) = \left[\frac{\varphi(1) - i\varphi(i)}{2}\right] z + \left[\frac{\varphi(1) + i\varphi(i)}{2}\right] \bar{z}$$

for all $z = x + iy \in \mathbb{C}$.

Date: September 26, 2009 (Version 1.0); due: October 2, 2009.

(iv) φ is given by

$$\varphi(x+iy) = (ax+by) + i(cx+dy)$$

for some $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathbb{R}^{2\times 2}$.

(c) Let $\varphi : \mathbb{C} \to \mathbb{C}$. Prove that the following statements are equivalent.

(i) φ is $\mathbb{C}$-linear.

(ii) φ is $\mathbb{R}$-linear and $\varphi(i) = i\varphi(1)$.

(iii) φ satisfies

$$\varphi(z) = \varphi(1)z$$

for all $z \in \mathbb{C}$.

(iv) φ is given by

$$\varphi(x+iy) = (ax-cy) + i(cx+ay)$$

for some $\left[\begin{smallmatrix} a & c \\ -c & a \end{smallmatrix}\right] \in \mathbb{R}^{2\times 2}$.

3. Let $\Omega \subseteq \mathbb{C}$ be a region and let $f : \Omega \to \mathbb{C}$. We will call f *complex differentiable* at $z \in \Omega$ if it is differentiable as defined in the lectures, ie. the limit

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} \tag{3.1}$$

exists. We will call f *real differentiable* at $z \in \Omega$ if there exists a $\mathbb{R}$-linear function $\varphi : \mathbb{C} \to \mathbb{C}$ such that

$$\lim_{h\to 0} \frac{f(z+h) - f(z) - \varphi(h)}{h} = 0. \tag{3.2}$$

(a) Prove that if f is complex differentiable at $z \in \Omega$, then f is real differentiable at z.

(b) Give an example to show that the converse of (a) is false.

(c) Let f be real differentiable at $z \in \Omega$. If the $\mathbb{R}$-linear function $\varphi : \mathbb{C} \to \mathbb{C}$ in (3.2) is also $\mathbb{C}$-linear, prove that f is complex differentiable at z. In this case, how is φ related to the limit in (3.1)?

(d) Let f be real differentiable at $z \in \Omega$. Show that if the limit

$$\lim_{h\to 0} \left| \frac{f(z+h) - f(z)}{h} \right| \tag{3.3}$$

exists[1], then either f or $\bar{f}$ must be complex differentiable at z. Give an example to show that f is not necessarily complex differentiable at z. Here the function $\bar{f} : \Omega \to \mathbb{C}$ is defined by $\bar{f}(z) = \overline{f(z)}$ for all $z \in \Omega$.

4. (a) Show that the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = \sqrt{|z^2 - \bar{z}^2|}$$

satisfies the Cauchy-Riemann equation at $z = 0$ but is not differentiable at $z = 0$.

(b) Let $\Omega \subseteq \mathbb{C}$ be a region such that the function

$$f(x+iy) = |x^2 - y^2| + 2i|xy|$$

is analytic on Ω but is not analytic on any larger region Ω' containing Ω. Find all possible Ω with this property.

(c) Find constants $a, b, c \in \mathbb{R}$ such that the functions $f, g : \mathbb{C} \to \mathbb{C}$ defined by

$$f(x+iy) = x + ay + i(bx+cy),$$
$$g(x+iy) = \cos x(\cosh y + a\sinh y) + i\sin x(\cosh y + b\sinh y)$$

are analytic on $\mathbb{C}$.

[1] Note the difference between (3.1) and (3.3).

5. Let $\Omega \subseteq \mathbb{C}$ be a region. Let $f : \Omega \to \mathbb{C}$ be analytic and $u(x,y) = \operatorname{Re} f(x+iy)$, $v(x,y) = \operatorname{Im} f(x+iy)$.

(a) Suppose $u, v \in C^2(\Omega_{\mathbb{R}})$. Show that u and v are *harmonic functions*, i.e. solutions of the Laplace equation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0,$$

on $\Omega_{\mathbb{R}}$.

(b) Let $a \in \mathbb{R}$. Suppose f is analytic on $D(0,1)$. Which of the following can occur as the real or imaginary part of f?

$$x^2 - axy + y^2, \quad x^3 - x^2 + y^3, \quad x^2 + y^2 - 5x, \quad \frac{x^2 - y^2}{(x^2+y^2)^2}.$$

6. We may rewrite any complex function f of two real variables x and y as a function of z and $\overline{z}$ via

$$x = \frac{z + \overline{z}}{2}, \quad y = \frac{z - \overline{z}}{2i}.$$

(a) Considering z and $\overline{z}$ as independent variables, show that

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \quad \text{and} \quad \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

(b) Show that the Cauchy-Riemann equation may be expressed as

$$\frac{\partial f}{\partial \overline{z}} = 0.$$

This may be interpreted as saying that complex differentiable functions must be independent[2] of $\overline{z}$ and depend only on z.

(c) Which of the following complex functions of two real variables can be expressed in terms of a polynomial in $z = x + iy$?

$$f_1(x,y) = x^2 - y^2 - ixy, \quad f_2(x,y) = x^2 + y^2 - 2ixy.$$

[2] In fact you may also view this as a reason why there isn't a 'quarternion analysis' similar to complex analysis. For a quarternion $q = x + yi + zj + wk$, its quarternionic conjugate $\overline{q} = x - yi - zj - wk$ can always be expressed in terms of q:

$$\overline{q} = -\frac{1}{2}(q + iqi + jqj + kqk),$$

and so we don't have functions dependent on q but not on $\overline{q}$.