# THE CAYLEY-HAMILTON THEOREM VIA COMPLEX INTEGRATION

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Let $f$ be an analytic function on the plane and $A$ a square matrix of order $n$ whose eigenvalues are contained in the interior of a circle $K$ centered at the origin. The expression

$$f(A) = \int_K f(z)(z - A)^{-1}dz$$

has been widely used to calculate (or define) the value of $f$ at $A$. A special case of this formula provides us with a trivial proof of the Cayley-Hamilton theorem.

We need only the most elementary notions: the integral $\int_K f(z)dz$, where $f$ is a continuous complex valued function on $K$, and the formulas

$$\int_K (z - a)^n dz = \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{otherwise,} \end{cases}$$

where $a$ lies inside $K$, and $\int_K p(z)dz = 0$, where $p$ is any polynomial.

Now let $F$ be a matrix valued function continuous at least on $K$. We define $\int_K F(z)dz$ to be the matrix whose entries are given by

$$\left[\int_K F(z)dz\right]_{jk} = \int_K [F(z)]_{jk}dz,$$

i.e., the function $F$ is integrated entry-wise. The familiar properties of integration extend to matrix valued functions. In particular, for $F: \mathcal{C} \to \mathfrak{M}_n$ and $B \in \mathfrak{M}_n: \int_K BF(z)dz = B\int_K F(z)dz$. The computation is easy.

LEMMA. *If $A$ and $K$ are as specified above, then $\int_K (z-A)^{-1}dz = 2\pi i I$.*

*Proof.* Note that $(z-A)^{-1} = \alpha(z)^{-1}\operatorname{adj}(z-A)$, where $\alpha$ is the characteristic polynomial of $A$ and adj is the classical adjoint (by adj $B$ we mean the matrix such that $B(\operatorname{adj} B) = (\det B)I$). Since the entries in $\operatorname{adj}(z-A)$ are polynomials in $z$, the entries in $(z-A)^{-1}$ are rational functions. Let $p(z)/\alpha(z)$ be such an entry. We have the partial fraction decomposition

$$p(z)/\alpha(z) = \sum_j b_j/(z - a_j) + R(z), \tag{*}$$

where the $a_j$'s are simple eigenvalues and $R(z)$ is a sum of terms $b/(z-a)^r$, $r > 1$. If $p(z)/\alpha(z)$ is off the diagonal in $(z-A)^{-1}$, the degree of $p$ is at most $n-2$, whereas that of $\alpha$ is $n$. Thus, to maintain the identity (*), we must have $\sum b_j$

$=0$. On the other hand, if $p(z)/\alpha(z)$ is a term on the diagonal of $(z-A)^{-1}$, $\deg p=n-1$, and we reason that $\sum b_j=1$. Thus

$$\begin{aligned}\int_K [(z-A)^{-1}]_{jk}dz &= \sum_\lambda \int_K b_\lambda(z-a_\lambda)^{-1}dz + \int_K R(z)dz \\ &= 2\pi i\Big(\sum_\lambda b_\lambda\Big) + 0 \\ &= 2\pi i\cdot\delta_{jk},\end{aligned}$$

and the Lemma is proved.

CAYLEY-HAMILTON THEOREM. *If $\alpha$ is the characteristic polynomial of the square matrix $A$, then $\alpha(A)=Z$, the zero matrix.*

*Proof.* Consider the matrix function

$$F(z) = (z^m - A^m)(z-A)^{-1} = z^{m-1} + z^{m-2}A + \cdots + A^{m-1}.$$

Since the entries in $F$ are clearly polynomials, $\int_K F(z)dz=Z$, and thus for $m=0, 1, 2, \cdots,$

$$\int_K z^m(z-A)^{-1}dz = \int_K A^m(a-A)^{-1}dz = A^m\cdot 2\pi i\cdot I.$$

It follows that

$$\begin{aligned}\alpha(A) &= \sum_{j=0}^n c_jA^j = \sum_{j=0}^n c_j\cdot\frac{1}{2\pi i}\int_K z^j(z-A)^{-1}dz \\ &= \frac{1}{2\pi i}\int_K\Big(\sum_{j=1}^n c_jz^j\Big)(z-A)^{-1}dz \\ &= \frac{1}{2\pi i}\int_K \alpha(z)(z-A)^{-1}dz \\ &= \frac{1}{2\pi i}\int_K \operatorname{adj}(z-A)dz \\ &= Z,\end{aligned}$$

since the entries of $\operatorname{adj}(z-A)$ are polynomials in $Z$.

A first course in complex variables is hardly a prerequisite for linear algebra; so the proof presented would be inappropriate for the latter subject. It does seem to be an uncommon application of integration theory, however. One might also note that the Fundamental Theorem of Algebra is commonly invoked in the study of eigenvalues, while its proof is usually left to complex analysis.