ON THE DEFINITION OF AN ANALYTIC FUNCTION

MAYNARD G. ARSOVE, University of Washington

1. Introduction. From the point of view of facility in developing the theory it is convenient to call a complex function f *analytic* on an open set Ω provided the derivative f' exists and is continuous on Ω. However, the continuity requirement on f' becomes superfluous in the light of the Cauchy-Goursat theorem, and for this reason the latter is often regarded as essential to a first course in complex function theory.

It is nevertheless true that conditions far weaker than the existence of f' will suffice to ensure analyticity. Theorems of Besicovitch, Looman-Menchoff, Maker and others* present such conditions, but their proofs draw on real function theory to such an extent that they are not generally considered germane to an introductory treatment of complex analysis.

As a possible replacement for the Cauchy-Goursat theorem we suggest the following criterion for analyticity, very closely related to results of Besicovitch [2] and Looman [3]. It appears, in fact, as a special case of the more elaborate theorem of Looman-Menchoff-Saks-Besicovitch-Maker [4, pp. 266–267]. Although the elementary nature of the proof which we submit is due largely to an observation of Meier [5] (stated explicitly in §3), the most complicated part of Meier's argument is avoided by the present hypotheses on f.

THEOREM. *Let $f=u+iv$ be a continuous complex function defined on an open subset Ω of the plane. If*

(1) $$\limsup_{\zeta\to z}\left|\frac{f(\zeta)-f(z)}{\zeta-z}\right|$$

is finite for all except perhaps a countable number of points z of Ω, and

(2) *the Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\qquad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

hold almost everywhere,

then f is analytic on Ω.

In the proof it has not been possible to dispense altogether with material belonging properly to the domain of real function theory, but an attempt has been made to eliminate as much of this as possible. What remain are (i) the Lebesgue bounded convergence theorem and (ii) the following special case of the Baire category theorem:

LEMMA. *Let Ω be an open set and F a non-empty subset closed in Ω. If F is covered by sets F_n $(n=1, 2, \cdots)$ closed in Ω, then there exists a neighborhood $\omega(\subset\Omega)$ and an index N such that $F_N\supset F\cap\omega\neq 0$.*

* See [2], [3], [4], [5], and in particular [6, pp. 195–201].

Certainly (i) is a result which (if unfamiliar to the student) can be stated without proof, while (ii) is easily established by elementary methods.

2. The areal mean functions. It is convenient to introduce the functions u_n and v_n obtained from u and v, respectively, by averaging over squares of side $1/n$:

$$
\begin{aligned}
u_n(x, y) &= n^2 \int_0^{1/n} \int_0^{1/n} u(x+s, y+t)\,ds\,dt, \\
v_n(x, y) &= n^2 \int_0^{1/n} \int_0^{1/n} v(x+s, y+t)\,ds\,dt.
\end{aligned} \tag{1}
$$

These functions are defined on the open set Ω_n consisting of all points (x, y) for which the square $[x, x+1/n]\times[y, y+1/n]$ lies in Ω.

As is well known [6, pp. 178–179], the continuity of u and v results in continuous differentiability of u_n and v_n. In fact, a direct calculation yields

$$
\begin{aligned}
\frac{\partial u_n(x, y)}{\partial x} &= n^2 \int_0^{1/n} [u(x+1/n, y+t) - u(x, y+t)]\,dt, \\
\frac{\partial u_n(x, y)}{\partial y} &= n^2 \int_0^{1/n} [u(x+s, y+1/n) - u(x+s, y)]\,ds,
\end{aligned}
$$

and $\partial v_n/\partial x$, $\partial v_n/\partial y$ are given analogously.

The motivation for considering the areal mean functions now becomes manifest with the observation that the theorem would follow at once if we could differentiate under the integral sign in (1). That is, the C' functions u_n and v_n would satisfy the Cauchy-Riemann equations, and the functions $f_n = u_n + iv_n$ would therefore be analytic; but, these functions clearly converge uniformly on compact subsets of Ω to f, so that f would necessarily be analytic on Ω.

However, there is no justification *a priori*, for differentiating under the integral sign. To remedy this deficiency, we have recourse to an argument based on the lemma of §1.

3. Proof of the theorem. Let F be the set of all points z of Ω such that f is not analytic on any neighborhood of z. Clearly, F is closed in Ω, and f is analytic on $\Omega - F$. We shall assume that F is not empty and show that this leads to a contradiction.

For positive integral n let F_n be the set of all points z of Ω such that

$$|\zeta - z| < 1/n \quad \text{implies} \quad |f(\zeta) - f(z)| \leqq n|\zeta - z|.$$

It is obvious from the continuity of f that each F_n is closed in Ω. Also, every point z at which $\limsup_{\zeta\to z} |f(\zeta)-f(z)|/|\zeta-z|$ is finite belongs to some F_n, and the remaining points of Ω are contained in a set of the form $\bigcup_{n=1}^{\infty} E_n$, where each E_n consists of a single point.

Inasmuch as the sets F_n and E_n in their aggregate cover Ω, and *a fortiori* F,

there exists according to the lemma a neighborhood ω intersecting F in a non-empty subset of one of the covering sets. This cannot occur for any of the covering sets E_n, in view of the fact that a continuous function analytic on a deleted neighborhood must actually be analytic on the full neighborhood. Hence, for some index N, $F_N \supset F \cap \omega$.

There is no loss of generality in supposing that the center z_0 of ω belongs to F and that the radius r_0 of ω is less than $1/N$. At every point z of $F_N \cap \omega$ the inequality

$$|f(\zeta) - f(z)| \leqq N|\zeta - z|$$

then holds for all ζ on ω. The following reasoning, due to Meier [5, p. 186], shows that this gives rise to the Lipschitz condition

$$|f(\zeta) - f(z)| \leqq 2N|\zeta - z| \tag{2}$$

for all ζ, z on the neighborhood ω' of radius $r_0/2$ about z_0.

First of all, we note that the inequality $|f'(z)| \leqq 2N$ is satisfied at all points z of $\omega' - F$. This follows from the Cauchy integral formula,* since z_0 in F implies that some point z' of F lies on the boundary β of the circle of analyticity of f about z and for ρ the radius of this circle

$$|f'(z)| = \left| \frac{1}{2\pi i} \oint_\beta \frac{f(\zeta) - f(z')}{(\zeta - z)^2} d\zeta \right| \leqq \frac{1}{2\pi} \cdot \frac{2N\rho}{\rho^2} \cdot 2\pi\rho = 2N.$$

Now, if the line segment $\overline{\zeta, z}$ joining ζ and z lies in $\omega' - F$, an integration along this segment yields (2) directly. On the other hand, if $\overline{\zeta, z}$ intersects F in a point z', then

$$\begin{aligned} |f(\zeta) - f(z)| &\leqq |f(\zeta) - f(z')| + |f(z') - f(z)| \\ &\leqq N|\zeta - z'| + N|z' - z| = N|\zeta - z|, \end{aligned}$$

and (2) is verified in all cases.

At this point we invoke the areal mean functions of §2, but for the region ω'. Taking (x, y) on ω_n' and $h(\neq 0)$ sufficiently close to 0, we find

$$\begin{aligned} \frac{u_n(x+h, y) - u_n(x, y)}{h} &- \frac{v_n(x, y+h) - v_n(x, y)}{h} \\ &= n^2 \int_0^{1/n} \int_0^{1/n} \left[\frac{u(x+h+s, y+t) - u(x+s, y+t)}{h} \right. \\ &\quad \left. - \frac{v(x+s, y+h+t) - v(x+s, y+t)}{h} \right] ds dt. \end{aligned}$$

In the present situation the Lipschitz condition (2) ensures that the absolute value of the integrand cannot exceed $4N$, and this permits us to employ

* A simple limiting case of the weak form of the Cauchy integral formula will suffice here.

the Lebesgue bounded convergence theorem. Since the integrand, by hypothesis, converges to 0 almost everywhere on $[0, 1/n]\times[0, 1/n]$ as $h\to 0$, we obtain

$$\frac{\partial u_n}{\partial x} = \frac{\partial v_n}{\partial y} \quad \text{on} \quad \omega_n'.$$

Similarly

$$\frac{\partial u_n}{\partial y} = -\frac{\partial v_n}{\partial x} \quad \text{on} \quad \omega_n'.$$

Thus, for each n the function $f_n = u_n + iv_n$ is analytic on ω_n'. These functions, moreover, converge uniformly to f on compact subsets of ω' and thereby confer their analyticity on f. Having arrived in this fashion at an obvious contradiction to the fact that z_0 belongs to F, we conclude that f must be analytic throughout Ω.

4. Essentially the same methods can be used to establish the following variant of Theorem 1 of Besicovitch [2].

THEOREM. *Let Ω be an open subset of the plane and $E(\subset\Omega)$ the union of countably many sets of zero length closed in Ω. If $f=u+iv$ is a bounded complex function on $\Omega-E$ such that*

(1) $\limsup_{\zeta\to z}\left|\dfrac{f(\zeta)-f(z)}{\zeta-z}\right|$ *is finite for z on $\Omega-E$, and*

(2) *the Cauchy-Riemann equations $u_x=v_y$, $u_y=-v_x$ hold almost everywhere on $\Omega-E$,*

then f can be extended so as to be analytic on Ω.

A discussion and proof of this result are given in [1].

Bibliography

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