MATH 114: GALOIS THEORY
SPRING 2008/09
PROBLEM SET 8

Let $p \in \mathbb{P}$ throughout this problem set. Let $n \in \mathbb{N}$. We will write $\mathbb{F}_{p^n}$ for the finite field of p^n elements constructed in Theorem **21** in the lectures.

1. The following are to be performed only with a straightedge and a compass. Prove your answers.
(a) Is it possible to trisect an angle of $90°$?
(b) Is it possible to construct a square whose area equals that of a circle of unit radius?
(c) Is it possible to construct the roots of $ax^2 + bx + c$ where a, b, c are constructible numbers?

2. Show that if E is a finite field, then $E \cong \mathbb{F}_{p^n}$ for some $p \in \mathbb{P}$ and $n \in \mathbb{N}$.

3. Let α and $\beta \in \overline{\mathbb{F}}_2$ be zeroes of $x^3 + x^2 + 1$ and $x^3 + x + 1 \in \mathbb{F}_2[x]$ respectively. Show that $\mathbb{F}_2(\alpha) = \mathbb{F}_2(\beta)$.

4. Show that every irreducible polynomial in $\mathbb{F}_p$ is a divisor of $x^{p^n} - x$ for some $n \in \mathbb{N}$.

5. Let K be a field, not necessarily finite. Show that if G is any finite multiplicative subgroup of the multiplicative group $K^\times$, then G is cyclic. In particular, this shows that $\mathbb{F}_{p^n}^\times$ is cyclic. [Hint: use the Fundamental Theorem of Finitely Generated Abelian Group from Math **113**.]

6. Let K be a field, not necessarily finite. Let $\mathrm{char}(K) = p \neq 0$.
(a) Show that the set $K^p = \{\alpha^p \mid \alpha \in K\}$ is a subfield of K.
(b) Let $K^p \leq L \leq K$. Show that if $[L : K^p] < \infty$, then $[L : K^p] = p^n$ for some $n \in \mathbb{N}$.

Date: April 25, 2009 (Version 1.1); due: May 1, 2009.