MATH 114: GALOIS THEORY
SPRING 2008/09
PROBLEM SET 6

If $K \leq \Omega$ is a field extension and $\alpha \in \Omega$ is algebraic over K, we will define the *degree* of α as $\deg f_K^\alpha(x)$, i.e. the degree of its irreducible polynomial over K.

1. Let $K \leq \Omega$ be a field extension.
(a) Let $a, b \in \Omega$ be algebraic over K. Prove that $a + b$ is algebraic over K.
(b) Suppose $a \in \Omega$ is algebraic over K of odd degree. Show that a^2 is also algebraic over K of odd degree and furthermore $K(a) = K(a^2)$.

2. Let $K \leq \Omega$ be a finite extension.
(a) Prove that if D is an integral domain and $K \subseteq D \subseteq \Omega$, then D is a field.
(b) Prove that if $[\Omega : K] \in \mathbb{P}$, then $\Omega = K(a)$ for any $a \in \Omega \backslash K$.
(c) Prove that if $f(x) \in K[x]$ is irreducible over K and $\deg f(x) \nmid [\Omega : K]$, then $f(x)$ has no zeroes in Ω. Hence or otherwise, show that $x^2 - 3$ is irreducible over $\mathbb{Q}(\sqrt[3]{2})$.

3. Let $K \leq L \leq M$ be a tower of field extensions. Theorem **6** in the lectures says that M is a *finite* extension over K iff M is a *finite* extension over L and L is a *finite* extension over K. Prove that the statement is still true if we replace *finite* by *algebraic* throughout.

4. Show that for any $a, b \in \mathbb{Q}$ such that $\sqrt{a} + \sqrt{b} \neq 0$, we must have $\mathbb{Q}(\sqrt{a} + \sqrt{b}) = \mathbb{Q}(\sqrt{a}, \sqrt{b})$.

5. Let $\mathbb{Q} \leq K$ and $[K : \mathbb{Q}] = 2$.
(a) Prove that there exists a unique square free integer m such that $K \cong \mathbb{Q}(\sqrt{m})$.
(b) Show that there are infinitely many such fields K that are pairwise non-isomorphic.

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