If \( K \leq \Omega \) is a field extension and \( \alpha \in \Omega \) is algebraic over \( K \), we will define the \textit{degree} of \( \alpha \) as \( \deg f_K^*(x) \), i.e. the degree of its irreducible polynomial over \( K \).

1. Let \( K \leq \Omega \) be a field extension.
   (a) Let \( a, b \in \Omega \) be algebraic over \( K \). Prove that \( a + b \) is algebraic over \( K \).
   (b) Suppose \( a \in \Omega \) is algebraic over \( K \) of odd degree. Show that \( a^2 \) is also algebraic over \( K \) of odd degree and furthermore \( K(a) = K(a^2) \).

2. Let \( K \leq \Omega \) be a finite extension.
   (a) Prove that if \( D \) is an integral domain and \( K \subseteq D \subseteq \Omega \), then \( D \) is a field.
   (b) Prove that if \( [\Omega : K] \in \mathbb{P} \), then \( \Omega = K(a) \) for any \( a \in \Omega \setminus K \).
   (c) Prove that if \( f(x) \in K[x] \) is irreducible over \( K \) and \( \deg f(x) \nmid [\Omega : K] \), then \( f(x) \) has no zeroes in \( \Omega \). Hence or otherwise, show that \( x^2 - 3 \) is irreducible over \( \mathbb{Q}(\sqrt[3]{2}) \).

3. Let \( K \leq L \leq M \) be a tower of field extensions. Theorem 6 in the lectures says that \( M \) is a \textit{finite} extension over \( K \) iff \( M \) is a \textit{finite} extension over \( L \) and \( L \) is a \textit{finite} extension over \( K \). Prove that the statement is still true if we replace \textit{finite} by \textit{algebraic} throughout.

4. Show that for any \( a, b \in \mathbb{Q} \) such that \( \sqrt{a} + \sqrt{b} \neq 0 \), we must have \( \mathbb{Q}(\sqrt{a} + \sqrt{b}) = \mathbb{Q}(\sqrt{a}, \sqrt{b}) \).

5. Let \( \mathbb{Q} \leq K \) and \( [K : \mathbb{Q}] = 2 \).
   (a) Prove that there exists a unique square free integer \( m \) such that \( K \cong \mathbb{Q}(\sqrt{m}) \).
   (b) Show that there are infinitely many such fields \( K \) that are pairwise non-isomorphic.