D will denote an integral domain throughout this problem set. \(D^*\) will denote the set of units of \(D\) and \(1_D\) will denote the unity in \(D\). Let \(\mathbb{P} = \{2, 3, 5, 7, \ldots \} \) denote the set of primes in \(\mathbb{Z}\).

1. Suppose there exists a function \(N : D \to \mathbb{Z}\) satisfying (i) \(N(a) = 0\) iff \(a = 0\), and (ii) \(N(ab) = N(a)N(b)\) for all \(a, b \in D\). Such a function is called a multiplicative norm on \(D\).
   (a) Prove that \(N(1_D) = 1\).
   (b) Prove that \(N(a) = \pm 1\) if \(a \in D^*\). Show that the converse is not always true.
   (c) Suppose \(D^* = \{a \in D \mid N(a) = \pm 1\}\). Show that if \(|N(a)| \in \mathbb{P}\), then \(a\) is irreducible.
   (d) Give an example where \(N\) defines a valuation on a Euclidean domain and another example where it doesn’t.

2. Consider the ring of Gaussian integers \(\mathbb{Z}[i] = \{m + ni \mid m, n \in \mathbb{Z}\}\). Observe that \(N(m + ni) = m^2 + n^2\) defines a multiplicative norm on \(\mathbb{Z}[i]\). Let \(p \in \mathbb{P}\).
   (a) Is \(p\) irreducible in \(\mathbb{Z}\)? Is \(p\) irreducible in \(\mathbb{Z}[i]\)?
   (b) Show that \(2 = ua^2\) where \(u \in \mathbb{Z}[i]^*\) and \(a \in \mathbb{Z}[i]\) is irreducible.
   (c) Show that if \(p \equiv 1 \mod 4\), then \(p\) is reducible in \(\mathbb{Z}[i]\).
   (d) Show that if \(p \equiv 3 \mod 4\), then \(p\) is irreducible in \(\mathbb{Z}[i]\).

3. Let \(k \in \mathbb{Z}\) be a square free integer, i.e. \(p^2 \nmid k\) for all \(p \in \mathbb{P}\). Let \(\mathbb{Z}[^2] := \{m + n\sqrt{k} \mid m, n \in \mathbb{Z}\}\). We will also assume \(k \neq 1\).
   (a) Show that \(N(m + n\sqrt{k}) = m^2 - kn^2\) defines a multiplicative norm on \(\mathbb{Z}[^2]\). (So a multiplicative norm can be negative valued, unlike norms on vector spaces or valuations on Euclidean domains).
   (b) Show that \(\mathbb{Z}[^2]^* = \{a \in \mathbb{Z}[^2] \mid |N(a)| = 1\}\).
   (c) Show that every non-zero, non-unit \(a \in \mathbb{Z}[^2]\) can be factored into a product of irreducible elements in \(\mathbb{Z}[^2]\).
   (d) Why doesn’t (c) contradict what we’ve proved in the lectures — that \(\mathbb{Z}[^{-5}]\) is not a unique factorization domain?

4. Let \(R\) be a ring. Show that the following three conditions are equivalent.
   **ACC:** Ascending chain condition: Every strictly increasing sequence of ideals is of finite length, i.e. \(I_1 \subset I_2 \subset \cdots \subset R \Rightarrow \) there exists \(N \in \mathbb{N}\) such that \(I_n = I_N\) for all \(n \geq N\).
   **MAC:** Maximum condition: Every nonempty set of ideals contains an ideal not properly contained in any other ideal of the set, i.e. if \(\emptyset \neq \mathcal{S} = \{I_\alpha \subset R \mid \alpha \in A\}\), then there exists some \(I_\beta \in \mathcal{S}\) such that \(I_\beta \nsubseteq I_\alpha\) for all \(\alpha \neq \beta\).
   **FBC:** Finite basis condition: Every ideal is finitely generated, i.e. if \(I \subsetneq R\), then there exists \(a_1, \ldots, a_n \in I\) such that \(I = \langle a_1, \ldots, a_n \rangle\).

In the lectures, we defined a Noetherian domain as one satisfying the ACC. Now we see that we could have used any of these three conditions. Note that there is no need to assume that \(R\) is commutative or has a unity to prove the equivalence of these conditions.

\textit{Date:} February 26, 2009 (Version 1.1); due: March 4, 2009.