MATH 114: GALOIS THEORY
SPRING 2008/09
PROBLEM SET 1

Throughout the problem set, unless stated otherwise, a *ring* will mean a commutative ring with unity 1 and will be denoted by R. We will assume that $0 \neq 1$ in R and will write R^* for the set of units. $\mathbb{Z}_n$ will denote the ring of integers modulo n. The ideal generated by a set $S \subseteq R$ will be denoted by $\langle S \rangle$.

1. Show that the following statements are equivalent:
i. R is a field;
ii. the only ideals in R are $\langle 0 \rangle$ and $\langle 1 \rangle$;
iii. for any ring R', a homomorphism $\varphi : R \to R'$ is injective.

2. We say that $a \in R$ is *nilpotent* if $a^n = 0$ for some $n \in \mathbb{N}$.
(a) Show that the set of all nilpotent elements in R is an ideal. We will write $N(R)$ for this set. It is called the *nilradical* of R.
(b) Find $N(\mathbb{Z})$, $N(\mathbb{Z}_{12})$, and $N(\mathbb{Z}_{32})$.
(c) Show that $N(R/N(R)) = \{0 + N(R)\}$ (note that this set has only one element, namely, the coset $0 + N(R)$). In other words, $R/N(R)$ has no nilpotent elements other than its zero element.

3. Suppose R satisfies the following properties
i. R has only one maximal ideal M;
ii. $R^* = \{1\}$.
Show that $R \cong \mathbb{Z}_2$.

4. If R has only one maximal ideal, then R is called a *local ring*.
(a) Show that every field is a local ring.
(b) Suppose $M \trianglelefteq R$, $M \neq R$ has the property that $a \notin M$ implies $a \in R^*$. Show that M is a maximal ideal and R is a local ring.
(c) Suppose M is a maximal ideal of R and has the property that $a \in 1 + M$ implies $a \in R^*$. Show that R is a local ring.

5. (a) Find all the ideals of $\mathbb{Q}$.
(b) Find all the subrings of $\mathbb{Q}$ (we require that a subring contains the unity).
(c) Can $\mathbb{Q}/\mathbb{Z}$, regarded as an additive quotient group, be a commutative ring with unity?

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