MATH 110: LINEAR ALGEBRA
SPRING 2007/08
PROBLEM SET 8

V denotes a finite dimensional vector space. If $\mathcal{T} \in \operatorname{End}(V)$, we will write $\mathcal{T}^2 = \mathcal{T} \circ \mathcal{T}, \mathcal{T}^3 = \mathcal{T} \circ \mathcal{T} \circ \mathcal{T}$, etc. We let $\mathcal{O} \in \operatorname{End}(V)$ and $\mathcal{I} \in \operatorname{End}(V)$ denote the zero and identity operators, ie. $\mathcal{O}(\mathbf{v}) = \mathbf{0}_V$ and $\mathcal{I}(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in V$.

1. Let V be a vector space over $\mathbb{F}$ and $\mathcal{S}, \mathcal{T} \in \operatorname{End}(V)$.
(a) Show that $\mathcal{I} - \mathcal{S} \circ \mathcal{T}$ is injective iff $\mathcal{I} - \mathcal{T} \circ \mathcal{S}$ is injective.
(b) $\mathcal{T}$ is called *nilpotent* if
$$\mathcal{T}^n = \mathcal{O}$$
for some $n \in \mathbb{N}$. Show that if $\mathcal{T}$ is nilpotent, then $\mathcal{I} - \mathcal{T}$ is bijective. What is $(\mathcal{I} - \mathcal{T})^{-1}$?

2. Let V be a vector space over $\mathbb{R}$ and $\mathcal{T} \in \operatorname{End}(V)$ be an involution, ie. $\mathcal{T}^2 = \mathcal{I}$. Define
$$V_+ := \{\mathbf{v} \in V \mid \mathcal{T}(\mathbf{v}) = \mathbf{v}\} \quad \text{and} \quad V_- := \{\mathbf{v} \in V \mid \mathcal{T}(\mathbf{v}) = -\mathbf{v}\}.$$
(a) Show that V_+ and V_- are subspaces of V.
(b) Show that
$$V_+ \oplus V_- = V.$$

3. Let $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$.
(a) Show that
$$\operatorname{nullsp}(A^\top A) = \operatorname{nullsp}(A).$$
(b) Show that
$$\operatorname{colsp}(A^\top A) = \operatorname{colsp}(A^\top).$$
(c) Deduce that
$$A^\top A\mathbf{x} = A^\top \mathbf{b}$$
always has a solution (even if $A\mathbf{x} = \mathbf{b}$ has no solution).
(d) Show that (a), (b), and (c) are false in general over arbitrary fields, ie. for $A \in \mathbb{F}^{m \times n}$ and $\mathbf{b} \in \mathbb{F}^m$.

4. Let V be a vector space over $\mathbb{R}$ and let $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ be an inner product. Let S be a subset (not necessarily a subspace) of V. We define the *orthogonal annihilator* of S, denoted $S^\perp$, to be the set
$$S^\perp = \{\mathbf{v} \in V \mid \langle \mathbf{v}, \mathbf{w} \rangle = 0 \text{ for all } \mathbf{w} \in S\}.$$
(a) Show that $S^\perp$ is always a subspace of V.
(b) Show that $\operatorname{span}(S) \subseteq (S^\perp)^\perp$.
(c) Show that if S_1 and S_2 are subsets of V and $S_1 \subseteq S_2$, then $S_2^\perp \subseteq S_1^\perp$.
(d) Show that $((S^\perp)^\perp)^\perp = S^\perp$.
(e) Show that either $S \cap S^\perp$ must be either the empty set $\varnothing$ or the zero subspace $\{\mathbf{0}_V\}$.
(f) Show that if W is a subspace, then $V = W \oplus W^\perp$. This is called the orthogonal complement of W.
(g) Show that if W is a subspace, then $W = (W^\perp)^\perp$.

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5. Let $(V, \langle \cdot\,,\,\cdot \rangle)$ be an inner product space over $\mathbb{R}$. Let W be a non-trivial subspace of V. By Problem **4**(f), we have that

$$V = W \oplus W^{\perp}.$$

Note that by Homework **2**, Problem **3**, this means that every $\mathbf{v} \in V$ can be written *uniquely* as

$$\mathbf{v} = \mathbf{w} + \mathbf{w}' \tag{5.1}$$

where $\mathbf{w} \in W$ and $\mathbf{w}' \in W^{\perp}$. We will define a function $\mathcal{P} : V \to V$ by $\mathcal{P}(\mathbf{v}) = \mathbf{w}$ for every $\mathbf{v} \in V$ according to the decomposition in (5.1). $\mathcal{P}$ is called the *orthogonal projection* onto W.

(a) Show that $\mathcal{P} \in \mathrm{End}(V)$.
(b) Show that $\mathcal{P}$ is idempotent, ie. $\mathcal{P}^2 = \mathcal{P}$.
(c) Show that $\langle \mathcal{P}(\mathbf{v}), \mathbf{v} \rangle \geq 0$ for every $\mathbf{v} \in V$.
(d) Show that $\|\mathcal{P}(\mathbf{v})\| \leq \|\mathbf{v}\|$ for every $\mathbf{v} \in V$.
(e) Show that $\mathcal{I} - \mathcal{P}$ is the orthogonal projection onto $W^{\perp}$.
(f) Show that

$$\|\mathbf{v}\|^2 = \|\mathcal{P}(\mathbf{v})\|^2 + \|(\mathcal{I} - \mathcal{P})(\mathbf{v})\|^2.$$