V denotes a finite dimensional vector space. If \( T \in \text{End}(V) \), we will write \( T^2 = T \circ T, T^3 = T \circ T \circ T \), etc. We let \( O \in \text{End}(V) \) and \( I \in \text{End}(V) \) denote the zero and identity operators, ie. \( O(v) = 0_V \) and \( I(v) = v \) for all \( v \in V \).

1. Let \( V \) be a vector space over \( F \) and \( S, T \in \text{End}(V) \).
   (a) Show that \( I - S \circ T \) is injective iff \( I - T \circ S \) is injective.
   (b) \( T \) is called nilpotent if \( T^n = O \) for some \( n \in \mathbb{N} \). Show that if \( T \) is nilpotent, then \( I - T \) is bijective. What is \( (I - T)^{-1} \)?

2. Let \( V \) be a vector space over \( \mathbb{R} \) and \( T \in \text{End}(V) \) be an involution, ie. \( T^2 = I \). Define \( V_+ := \{ v \in V \mid T(v) = v \} \) and \( V_- := \{ v \in V \mid T(v) = -v \} \).
   (a) Show that \( V_+ \) and \( V_- \) are subspaces of \( V \).
   (b) Show that \( V_+ \oplus V_- = V \).

3. Let \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \).
   (a) Show that \( \text{nullsp}(A^\top A) = \text{nullsp}(A) \).
   (b) Show that \( \text{colsp}(A^\top A) = \text{colsp}(A^\top) \).
   (c) Deduce that \( A^\top A x = A^\top b \) always has a solution (even if \( A x = b \) has no solution).
   (d) Show that (a), (b), and (c) are false in general over arbitrary fields, ie. for \( A \in \mathbb{F}^{m \times n} \) and \( b \in \mathbb{F}^m \).

4. Let \( V \) be a vector space over \( \mathbb{R} \) and let \( (\cdot, \cdot) : V \times V \rightarrow \mathbb{R} \) be an inner product. Let \( S \) be a subset (not necessarily a subspace) of \( V \). We define the orthogonal annihilator of \( S \), denoted \( S^\perp \), to be the set \( S^\perp = \{ v \in V \mid \langle v, w \rangle = 0 \text{ for all } w \in S \} \).
   (a) Show that \( S^\perp \) is always a subspace of \( V \).
   (b) Show that \( \text{span}(S) \subseteq (S^\perp)^\perp \).
   (c) Show that if \( S_1 \) and \( S_2 \) are subsets of \( V \) and \( S_1 \subseteq S_2 \), then \( S_2^\perp \subseteq S_1^\perp \).
   (d) Show that \( ((S^\perp)^\perp)^\perp = S^\perp \).
   (e) Show that either \( S \cap S^\perp \) must be either the empty set \( \emptyset \) or the zero subspace \( \{0_V\} \).
   (f) Show that if \( W \) is a subspace, then \( V = W \oplus W^\perp \). This is called the orthogonal complement of \( W \).
   (g) Show that if \( W \) is a subspace, then \( W = (W^\perp)^\perp \).

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5. Let \((V, \langle \cdot, \cdot \rangle)\) be an inner product space over \(\mathbb{R}\). Let \(W\) be a non-trivial subspace of \(V\). By Problem 4(f), we have that
\[ V = W \oplus W^\perp. \]
Note that by Homework 2, Problem 3, this means that every \(v \in V\) can be written uniquely as
\[ v = w + w' \tag{5.1} \]
where \(w \in W\) and \(w' \in W^\perp\). We will define a function \(P : V \rightarrow V\) by \(P(v) = w\) for every \(v \in V\) according to the decomposition in (5.1). \(P\) is called the orthogonal projection onto \(W\).
(a) Show that \(P \in \text{End}(V)\).
(b) Show that \(P\) is idempotent, i.e. \(P^2 = P\).
(c) Show that \(\langle P(v), v \rangle \geq 0\) for every \(v \in V\).
(d) Show that \(\|P(v)\| \leq \|v\|\) for every \(v \in V\).
(e) Show that \(I - P\) is the orthogonal projection onto \(W^\perp\).
(f) Show that
\[ \|v\|^2 = \|P(v)\|^2 + \|(I - P)(v)\|^2. \]