1. Let \( A, B \in \mathbb{F}^{n \times n} \). Define the function \( T : \mathbb{F}^{n \times n} \to \mathbb{F}^{n \times n} \) by
\[
T(X) = AXB
\]
for all \( X \in \mathbb{F}^{n \times n} \).
(a) Show that \( T \in \text{End}(\mathbb{F}^{n \times n}) \).
SOLUTION. Let \( \alpha_1, \alpha_2 \in \mathbb{F} \) and \( X_1, X_2 \in \mathbb{F}^{n \times n} \). Then by the distributive property of matrix multiplication,
\[
T(\alpha_1X_1 + \alpha_2X_2) = \alpha_1AX_1B + \alpha_2AX_2B
\]
Hence \( T \) is linear.

(b) Show that \( T \) is invertible if and only if \( A \) and \( B \) are nonsingular matrices.
SOLUTION. Suppose \( A \) and \( B \) are nonsingular. Define the map \( S : \mathbb{F}^{n \times n} \to \mathbb{F}^{n \times n} \) by
\[
S(X) = A^{-1}X(AB)^{-1}
\]
for all \( X \in \mathbb{F}^{n \times n} \). By the associativity of matrix multiplication, we have
\[
S(T(X)) = A^{-1}(AXB)B^{-1} = (A^{-1}A)(B^{-1}B) = IXI = X,
\]
\[
T(S(X)) = A(A^{-1}X(AB)^{-1})B = (AA^{-1})X(B^{-1}B) = IXI = X
\]
for all \( X \in \mathbb{F}^{n \times n} \). So \( S \circ T = I = T \circ S \) and so \( T^{-1} = S \). Suppose \( T \) is invertible, then \( \ker(T) = \{O\} \). We will prove by contradiction. Suppose \( A \) or \( B \) is singular. Without loss of generality, we may assume that \( A \) is singular (the argument for singular \( B \) is similarly). Then there exists a non-zero \( x \in \text{nullsp}(A) \), ie. \( Ax = 0 \) but \( x \neq 0 \). Define the matrix \( X \in \mathbb{F}^{n \times n} \) all of whose columns are \( x \), ie.
\[
X = [x, \ldots, x].
\]
Then \( X \neq O \) but
\[
T(X) = AXB = A[x, \ldots, x]B = [Ax, \ldots, Ax]B = [0, \ldots, 0]B = OB = O.
\]
So \( \ker(T) \neq \{O\} \) and so \( T \) is not invertible.

2. Let \( V \) be a finite dimensional vector space and \( T \in \text{End}(V) \). Let \( \dim(V) = n \). Let \( v \in V \) be such that
\[
T^{n-1}(v) \neq 0 \quad \text{and} \quad T^n(v) = 0.
\]
(a) Show that the vectors \( v, T(v), T^2(v), \ldots, T^{n-1}(v) \) form a basis for \( V \).
SOLUTION. Let \( \alpha_0, \ldots, \alpha_{n-1} \in \mathbb{F} \) be such that
\[
\alpha_0v + \alpha_1T(v) + \alpha_2T^2(v) + \cdots + \alpha_{n-1}T^{n-1}(v) = 0.
\]
(2.1)
Applying $T^{n-1}$ to both sides of (2.1), we obtain
\[ T^{n-1}(\alpha_0 v + \alpha_1 T(v) + \alpha_2 T^2(v) + \cdots + \alpha_{n-1} T^{n-1}(v)) = T^{n-1}(0), \]
\[ \alpha_0 T^{n-1}(v) + \alpha_1 T^n(v) + \alpha_2 T^{n+1}(v) + \cdots + \alpha_{n-1} T^{2n-2}(v) = 0. \]
Note that for all $m \geq n$, $T^m(v) = T^{m-n}(T^n(v)) = T^{m-n}(0) = 0$. So the last equation becomes
\[ \alpha_0 T^{n-1}(v) = 0. \]
Since $T^{n-1}(v) \neq 0$, we must have $\alpha_0 = 0$. Now, (2.1) becomes
\[ \alpha_1 T(v) + \alpha_2 T^2(v) + \cdots + \alpha_{n-1} T^{n-1}(v) = 0 \]
and we apply $T^{n-2}$ to both sides and use the same argument above to conclude that $\alpha_1 = 0$. Repeating this argument $n$ times gives
\[ \alpha_0 = \alpha_1 = \cdots = \alpha_{n-1} = 0. \]
Hence $\{v, T(v), T^2(v), \ldots, T^{n-1}(v)\}$ is linearly independent and since $\dim(V) = n$, it forms a basis of $V$.

(b) Let $B$ be the basis in (a). What is the matrix representation $[T]_{B,B}$?

SOLUTION. We apply $T$ to each vector of $B$ in turn to get
\[ T(v) = 0v + 1T(v) + 0T^2(v) + \cdots + 0T^{n-1}(v), \]
\[ T(T(v)) = 0v + 0T(v) + 1T^2(v) + \cdots + 0T^{n-1}(v), \]
\[ \vdots \]
\[ T(T^{n-2}(v)) = 0v + 0T(v) + 0T^2(v) + \cdots + 1T^{n-1}(v), \]
\[ T(T^{n-1}(v)) = 0v + 0T(v) + 0T^2(v) + \cdots + 0T^{n-1}(v). \]
Hence writing the coefficients as *columns* yield the required matrix representation
\[
[T]_{B,B} = \begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1 & 0 \\
0 & 0 & \cdots & \cdots & 0 & 0 & 1 \\
0 & 0 & \cdots & \cdots & 0 & 0 & 0
\end{bmatrix},
\]
a matrix with 1’s on the super-diagonal the 0’s everywhere else.

3. Let $V$ and $W$ be finite-dimensional vector spaces over $F$.

(a) Let $T \in \text{Hom}(V,W)$. Prove the following.
   (i) If $T$ is injective, then $\dim(V) \leq \dim(W)$.
   (ii) If $T$ is surjective, then $\dim(V) \geq \dim(W)$.
   (iii) If $T$ is bijective, then $\dim(V) = \dim(W)$.

SOLUTION. Since $\text{im}(T)$ is a subspace of $W$, we must have
\[ \dim(W) \geq \dim(\text{im}(T)) = \text{rank}(T) = \dim(V) - \text{nullity}(T). \quad (3.2) \]
If $T$ is injective, then $\text{nullity}(T) = 0$ and so
\[ \dim(W) \geq \dim(V). \]
If $T$ is surjective, then $W = \text{im}(T)$ and equality holds in (3.2), ie.
\[ \dim(W) = \dim(V) - \text{nullity}(T). \]
Hence
\[ \dim(V) = \dim(W) + \text{nullity}(T) \geq \dim(W). \]

If \( T \) is bijective, then being both injective and surjective, we have \( \dim(V) \leq \dim(W) \) and \( \dim(V) \geq \dim(W) \) and so
\[ \dim(V) = \dim(W). \]

(b) Show that if \( \dim(V) = \dim(W) \), then there exists a bijective \( T \in \text{Hom}(V,W) \). [Together with (iii), this shows that ‘V and W are isomorphic if and only if \( \dim(V) = \dim(W) \).’]

SOLUTION. Let \( n = \dim(V) = \dim(W) \). Let \( \{v_1, \ldots, v_n\} \) and \( \{w_1, \ldots, w_n\} \) be bases of \( V \) and \( W \) respectively. We define \( T : V \to W \) by
\[ T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) = \alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_n w_n \]
for all \( \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{F} \) (note that every \( v \in V \) may be written in this form). \( T \) is linear since for \( \lambda, \mu \in \mathbb{F} \) and \( \alpha_1 v_1 + \cdots + \alpha_n v_n, \beta_1 v_1 + \cdots + \beta_n v_n \in V \),
\begin{align*}
T(\lambda(\alpha_1 v_1 + \cdots + \alpha_n v_n) + \mu(\beta_1 v_1 + \cdots + \beta_n v_n)) &= T((\lambda \alpha_1 + \mu \beta_1) v_1 + \cdots + (\lambda \alpha_n + \mu \beta_n) v_n) \\
&= (\lambda \alpha_1 + \mu \beta_1) w_1 + \cdots + (\lambda \alpha_n + \mu \beta_n) w_n \\
&= \lambda(\alpha_1 w_1 + \cdots + \alpha_n w_n) + \mu(\beta_1 w_1 + \cdots + \beta_n w_n) \\
&= \lambda T(\alpha_1 v_1 + \cdots + \alpha_n v_n) + \mu T(\beta_1 v_1 + \cdots + \beta_n v_n).
\end{align*}

Let \( \alpha_1 v_1 + \cdots + \alpha_n v_n \in \ker(T) \). Then
\[ 0_V = T(\alpha_1 v_1 + \cdots + \alpha_n v_n) = \alpha_1 w_1 + \cdots + \alpha_n w_n. \]

Since \( w_1, \ldots, w_n \) are linearly independent, we get \( \alpha_1 = \cdots = \alpha_n = 0 \) and so
\[ \alpha_1 v_1 + \cdots + \alpha_n v_n = 0 v_1 + \cdots + 0 v_n = 0_V. \]

Hence \( \ker(T) = \{0_V\} \) and so \( T \) is injective.

(c) Let \( \dim(V) = \dim(W) \). Let \( T \in \text{Hom}(V,W) \) and \( S \in \text{Hom}(W,V) \). Show that
\[ S \circ T = \mathcal{I}_V \quad (3.3) \]
if and only if
\[ T \circ S = \mathcal{I}_W. \]

SOLUTION. Let \( S \circ T = \mathcal{I}_V \). Then \( T \) is injective since if \( T(v) = 0_W \), then \( v = \mathcal{I}_V(v) = S(T(v)) = S(0_W) = 0_V \) (ie. \( \ker(T) = \{0_V\} \)). Since \( V \) is finite-dimensional, we may apply Theorem 4.12 to conclude that \( T \) is invertible. Let \( T^{-1} \) be the inverse of \( T \). Then composing \( T^{-1} \) on the right of (3.3), we get
\begin{align*}
(S \circ T) \circ T^{-1} &= \mathcal{I}_V \circ T^{-1} = T^{-1}, \\
S \circ (T \circ T^{-1}) &= T^{-1}, \\
S \circ \mathcal{I}_V &= T^{-1}, \\
S &= T^{-1}.
\end{align*}

Hence
\[ T \circ S = \mathcal{I}_W \]
as required. For the converse, just swap the roles of \( T \) and \( S \).

(d) Show that (b) and (c) are false if \( \dim(V) \neq \dim(W) \).

SOLUTION. If \( \dim(V) \neq \dim(W) \) and there exists a bijective \( T \), then this would contradict (iii) in (a).
4. Let \( \mathbb{P} \) be the vector space of all polynomials over \( \mathbb{R} \). Define the functions \( D : \mathbb{P} \to \mathbb{P} \) and \( M : \mathbb{P} \to \mathbb{P} \) by

\[
D(p)(x) = p'(x) \quad \text{and} \quad M(p)(x) = xp(x)
\]

for all \( p \in \mathbb{P} \), i.e. the ‘differentiation with respect to \( x \)’ and ‘multiplication by \( x \)’ functions. Explicitly, if \( p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_d x^d \) with \( a_0, a_1, a_2, \ldots, a_d \in \mathbb{R} \), then

\[
D(p)(x) = a_1 + 2a_2 x + \cdots + da_dx^{d-1},
\]

\[
M(p)(x) = a_0 x + a_1 x^2 + \cdots + a_d x^{d+1}.
\]

(a) Show that \( D \in \text{End}(\mathbb{P}) \) and \( M \in \text{End}(\mathbb{P}) \).

**SOLUTION.** Let \( \lambda, \mu \in \mathbb{F} \) and \( p(x) = a_0 + a_1 x + \cdots + a_d x^d, q(x) = b_0 + b_1 x + \cdots + b_d x^d \in \mathbb{P} \) where \( d = \max\{\deg(p(x)), \deg(q(x))\} \). Then

\[
D(\lambda p + \mu q)(x) = (\lambda a_1 + \mu b_1) + 2(\lambda a_2 + \mu b_2)x + \cdots + d(\lambda a_d + \mu b_d)x^{d-1}
\]

\[
= \lambda(a_1 + 2a_2 x + \cdots + da_dx^{d-1}) + \mu(b_1 + 2b_2 x + \cdots + db_dx^{d-1})
\]

\[
= \lambda D(p)(x) + \mu D(q)(x)
\]

and

\[
M(\lambda p + \mu q)(x) = (\lambda a_0 + \mu b_0)x + (\lambda a_1 + \mu b_1)x^2 + \cdots + (\lambda a_d + \mu b_d)x^{d+1}
\]

\[
= \lambda(a_0 x + a_1 x^2 + \cdots + a_dx^{d+1}) + \mu(b_0 x + b_1 x^2 + \cdots + b_dx^{d+1})
\]

\[
= \lambda M(p)(x) + \mu M(q)(x).
\]

So \( D \) and \( M \) are linear.

(b) Show that

\[
\text{im}(D) = \mathbb{P}, \quad \ker(D) \neq \{0(x)\}, \quad \text{im}(M) \neq \mathbb{P}, \quad \ker(M) = \{0(x)\},
\]

where \( 0(x) \) denotes the zero polynomial.

**SOLUTION.** Let \( p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_d x^d \in \mathbb{P} \). If we let

\[
q(x) = a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \cdots + \frac{a_d}{d+1} x^{d+1},
\]

we see that

\[
D(q)(x) = p(x).
\]

So \( D \) is surjective and \( \text{im}(D) = \mathbb{P} \). Let \( c(x) = 1 \). Then \( D(c)(x) = 0(x) \) and so \( c(x) \in \ker(D) \). Since \( c(x) \neq 0(x) \), \( \ker(D) \neq \{0(x)\} \). Let \( p(x) \in \ker(M) \), then \( M(p)(x) = 0(x) \), i.e.

\[
a_0 x + a_1 x^2 + \cdots + a_d x^{d+1} = 0x + 0x^2 + \cdots + 0x^{d+1}.
\]

So \( a_0 = a_1 = \cdots = a_d = 0 \) and so \( p(x) = 0(x) \). Note that \( c(x) \notin \text{im}(M) \) since if

\[
M(p)(x) = c(x),
\]

then \( \deg(M(p)(x)) = \deg(c(x)) = 1 \), which is only possible of \( p(x) = 0(x) \) but clearly \( M(0)(x) = 0(x) \neq c(x) \).

(c) Are \( D \) and \( M \) surjective, injective, or bijective? Why would these observations not contradict Theorem 4.12 from the lectures?

**SOLUTION.** Theorem 4.12 applies only to finite-dimensional vector spaces whereas \( \mathbb{P} \) is infinite-dimensional.

(d) Show that

\[
D \circ M - M \circ D = I_\mathbb{P}
\]

and more generally

\[
D^n \circ M - M \circ D^n = nD^{n-1}
\]

for all \( n \in \mathbb{N} \).
SOLUTION. Let \( p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_d x^d \in \mathbb{P} \). Note that
\[
\mathcal{M}(\mathcal{D}(p))(x) = a_1 x + 2a_2 x^2 + \cdots + d a_d x^d, \\
\mathcal{D}(\mathcal{M}(p))(x) = a_0 + 2a_1 x + \cdots + (d+1) a_d x^d,
\]
and so
\[
(\mathcal{D} \circ \mathcal{M} - \mathcal{M} \circ \mathcal{D})(p)(x) = \mathcal{D}(\mathcal{M}(p))(x) - \mathcal{M}(\mathcal{D}(p))(x) \\
= a_0 + a_1 x + a_2 x^2 + \cdots + a_d x^d \\
= p(x).
\]
Hence \( \mathcal{D} \circ \mathcal{M} - \mathcal{M} \circ \mathcal{D} = I_\mathbb{P} \). For the general case, we will use induction. We have already shown that it is true for \( n = 1 \). Suppose it is true for all \( n = 1, \ldots, k \), we will use this to deduce that it is also true for \( n = k+1 \),
\[
\mathcal{D}^{k+1} \circ \mathcal{M} - \mathcal{M} \circ \mathcal{D}^{k+1} = \mathcal{D} \circ (\mathcal{D}^k \circ \mathcal{M}) - \mathcal{M} \circ \mathcal{D}^{k+1} \\
= \mathcal{D} \circ (k \mathcal{D}^{k-1} + \mathcal{M} \circ \mathcal{D}^k) - \mathcal{M} \circ \mathcal{D}^{k+1} \\
= k \mathcal{D}^k + \mathcal{D} \circ \mathcal{M} \circ \mathcal{D}^k - \mathcal{M} \circ \mathcal{D}^{k+1} \\
= k \mathcal{D}^k + (\mathcal{D} \circ \mathcal{M} - \mathcal{M} \circ \mathcal{D}) \circ \mathcal{D}^k \\
= k \mathcal{D}^k + I_\mathbb{P} \circ \mathcal{D}^k \\
= k \mathcal{D}^k + \mathcal{D}^k \\
= (k + 1) \mathcal{D}^k
\]
as required.