1. Let \( W_1, W_2 \) be subspaces of \( V \) such that \( V = W_1 \oplus W_2 \). Let \( W \) be a subspace of \( V \). Show that if \( W_1 \subseteq W \) or \( W_2 \subseteq W \), then
\[
W = (W \cap W_1) \oplus (W \cap W_2).
\]
Is this still true if we omit the condition ‘\( W_1 \subseteq W \) or \( W_2 \subseteq W \)’?

**Solution.** Without loss of generality, we will assume that \( W_1 \subseteq W \). So \( W \cap W_1 = W_1 \) and we just need to show that
\[
W = W_1 \oplus (W \cap W_2).
\]
By Problem 2(b) in Homework 2, the smallest subspace containing both \( W_1 \) and \( W \cap W_2 \) is \( W_1 + (W \cap W_2) \); since \( W_1 \subseteq W \) and \( W \cap W_2 \subseteq W \), it follows that
\[
W_1 + (W \cap W_2) \subseteq W. \tag{1.1}
\]
To show the reverse inclusion, let \( w \in W \). Then \( w \in V \). Since \( V = W_1 \oplus W_2 \), there exist \( w_1 \in W_1 \) and \( w_2 \in W_2 \) such that
\[
w = w_1 + w_2.
\]
Observe that
\[
w_2 = w - w_1 \in W \cap W_2
\]
and \( w - w_1 \in W \). Hence \( w = w_1 + w_2 \in W_1 + (W \cap W_2) \). Since this is true for arbitrary \( w \in W \), we conclude that
\[
W \subseteq W_1 + (W \cap W_2). \tag{1.2}
\]
By (1.1) and (1.2),
\[
W = W_1 + (W \cap W_2).
\]
This is a direct sum since
\[
(W \cap W_1) \cap (W \cap W_2) \subseteq W_1 \cap W_2 = \{0\}.
\]
The statement is false without the condition ‘\( W_1 \subseteq W \) or \( W_2 \subseteq W \)’. Here is a counter-example.
Let \( W = \{(x, y) \in \mathbb{R}^2 \mid x = y\} \), \( W_1 = \{(x, y) \in \mathbb{R}^2 \mid y = 0\} \), and \( W_2 = \{(x, y) \in \mathbb{R}^2 \mid x = 0\} \).
Then \( W_1 \oplus W_2 = \mathbb{R}^2 \) since every \( (x, y) \in \mathbb{R}^2 \) may be written uniquely as \( (x, y) = (x, 0) + (0, y) \).
However, \( W \cap W_1 = \{(0, 0)\} = W \cap W_2 \) and so
\[
(W \cap W_1) \oplus (W \cap W_2) = \{(0, 0)\} \neq W.
\]

2. For the following vector spaces \( V \), find the coordinate representation of the respective elements.
(a) \( V = \mathbb{P}_2 = \{ax^2 + bx + c \mid a, b, c \in \mathbb{R}\} \). Find \( [p(x)]_\mathcal{B} \) where
\[
p(x) = 2x^2 - 5x + 6, \quad \mathcal{B} = [1, x - 1, (x - 1)^2].
\]

**Solution.** Since
\[
p(x) = 2x^2 - 5x + 6 = 3 - (x - 1) + 2(x - 1)^2,
\]
so
\[
[p(x)]_\mathcal{B} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}.
\]

*Date: March 15, 2008 (Version 1.1).*
(b) \( V = \mathbb{R}^{2 \times 2} \). Find \([A]_{\mathcal{B}}\) where
\[
A = \begin{bmatrix} 2 & 3 \\ 4 & -7 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}.
\]

**SOLUTION.** Let
\[
\begin{bmatrix} 2 & 3 \\ 4 & -7 \end{bmatrix} = \alpha \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \gamma \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + \delta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.
\]
Then solving
\[
\begin{align*}
\alpha + \gamma + \delta &= 2, \\
\alpha - \beta - \gamma &= 3, \\
\alpha + \beta &= 4, \\
\alpha &= -7,
\end{align*}
\]
gives \( \alpha = -7, \beta = 11, \gamma = -21, \delta = 30 \). So
\[
[A]_{\mathcal{B}} = \begin{bmatrix} -7 \\ 11 \\ -21 \\ 30 \end{bmatrix}.
\]

(c) \( V = \mathbb{R}^2 \). Let \( \theta \in \mathbb{R} \) be fixed. Find \([v]_{\mathcal{B}}\) where
\[
v = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} \cos \theta \\ \sin \theta \\ -\sin \theta \\ \cos \theta \end{bmatrix}.
\]

**SOLUTION.** Let
\[
\begin{bmatrix} x \\ y \end{bmatrix} = \alpha \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} + \beta \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}.
\]
Then solving
\[
\begin{align*}
x &= \alpha \cos \theta - \beta \sin \theta, \\
y &= \alpha \sin \theta + \beta \cos \theta,
\end{align*}
\]
gives
\[
\begin{align*}
\alpha &= x \cos \theta + y \sin \theta, \\
\beta &= -x \sin \theta + y \cos \theta.
\end{align*}
\]
So
\[
[v]_{\mathcal{B}} = \begin{bmatrix} x \cos \theta + y \sin \theta \\ -x \sin \theta + y \cos \theta \end{bmatrix}.
\]

3. Let \( W_1 \) and \( W_2 \) be the following subspaces of \( \mathbb{R}^4 \).
\[
W_1 = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 6 \\ 4 \\ 3 \\ 6 \\ 5 \\ 1 \\ 6 \\ 1 \\ 12 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix} \right\}, \quad W_2 = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}.
\]

(a) Find a basis of \( W_1 \cap W_2 \).

**SOLUTION.** Let the three vectors spanning \( W_1 \) be \( u_1, u_2, u_3 \) and the two vectors spanning \( W_2 \) be \( v_1, v_2 \) respectively. If \( w \in W_1 \cap W_2 \), then there exist \( \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \) and \( \beta_1, \beta_2 \in \mathbb{R} \) such that
\[
\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = w = \beta_1 v_1 + \beta_2 v_2.
\]
In other words, we want to solve

\[
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}
+ \alpha_2 \begin{bmatrix}
4 \\
-1 \\
3
\end{bmatrix}
+ \alpha_3 \begin{bmatrix}
5 \\
1 \\
6
\end{bmatrix}
= \beta_1 \begin{bmatrix}
1 \\
-1 \\
1
\end{bmatrix}
+ \beta_2 \begin{bmatrix}
2 \\
-1 \\
4
\end{bmatrix}.
\]

Forming the augmented system to solve for \(\alpha_1, \alpha_2, \alpha_3\) in terms of \(\beta_1, \beta_2\) (alternatively, we could also solve for \(\beta_1\) and \(\beta_2\) in terms of \(\alpha_1, \alpha_2, \alpha_3\), we get upon Gauss-Jordan elimination,

\[
\begin{bmatrix}
1 & 4 & 5 \\
2 & -1 & 1 \\
3 & 3 & 6
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}.
\]

For this system to be consistent, we must have \(\beta_1 + 3\beta_2 = 0\). Hence

\[
w = \beta_1 \begin{bmatrix}
1 \\
-1 \\
1
\end{bmatrix}
+ \beta_2 \begin{bmatrix}
2 \\
-1 \\
4
\end{bmatrix}
= \beta_2 \begin{bmatrix}
2 \\
1 \\
2
\end{bmatrix}
\]

for some \(\beta_2 \in \mathbb{R}\). In other words,

\[
W_1 \cap W_2 = \text{span}\{[-1, 2, 1, 2]^\top\}.
\]

(b) Find a basis of \(W_1 + W_2\).

\textbf{SOLUTION.} We will do this in (e). For an alternative way, see the solution to Homework 5, Problem 1 from Fall 2007.

(c) Extend the basis of \(W_1 \cap W_2\) in (a) to get a basis of \(W_1\).

\textbf{SOLUTION.} We will apply the Adding-On Algorithm to \{[-1, 2, -1, 2]^\top\} and check each of the three vectors \(u_1, u_2, u_3\) in turn. Since

\[
\begin{bmatrix}
1 \\
-1 \\
1
\end{bmatrix} = \lambda \begin{bmatrix}
-1 \\
2 \\
-1
\end{bmatrix}
\]

clearly has no solution, we add \(u_1\) to get \{[-1, 2, 1, 2]^\top, [1, -1, 1, 1]^\top\}. Since

\[
\begin{bmatrix}
4 \\
-1 \\
3
\end{bmatrix} = \lambda_1 \begin{bmatrix}
-1 \\
2 \\
1
\end{bmatrix} + \lambda_2 \begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}
\]

has a solution \(\lambda_1 = -9/4, \lambda_2 = 7/4\), \(u_2\) need not be added. Since

\[
\begin{bmatrix}
5 \\
1 \\
6
\end{bmatrix} = \lambda_1 \begin{bmatrix}
-1 \\
2 \\
1
\end{bmatrix} + \lambda_2 \begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}
\]

has a solution \(\lambda_1 = -9/4, \lambda_2 = 11/4\), \(u_3\) need not be added. Hence \{[-1, 2, 1, 2]^\top, [1, 2, 3, 6]^\top\} is a basis for \(W_1\) by the Adding-On Algorithm.

(d) Extend the basis of \(W_1 \cap W_2\) in (a) to get a basis of \(W_2\).

\textbf{SOLUTION.} We will apply the Adding-On Algorithm to \{[-1, 2, -1, 2]^\top\} and check each of the two vectors \(v_1, v_2\) in turn. Since

\[
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix} = \lambda \begin{bmatrix}
-1 \\
2 \\
-1
\end{bmatrix}
\]

3
clearly has no solution, we add \( v_1 \) to get \( \{-1, 2, -1, 2\}^\top, [1, 2, 3, 6]^\top \}. Since
\[
\begin{bmatrix}
2 \\
-1 \\
4 \\
5
\end{bmatrix} = \lambda_1 \begin{bmatrix}
-1 \\
2 \\
1 \\
2
\end{bmatrix} + \lambda_2 \begin{bmatrix}
1 \\
-1 \\
1 \\
1
\end{bmatrix}
\]
has a solution \( \lambda_1 = 1, \lambda_2 = 3, v_2 \) need not be added. Hence \( \{-1, 2, 1, 2\}^\top, [1, -1, 1, 1]^\top \} \) is a basis for \( W_2 \) by the Adding-On Algorithm.

(c) From the bases in (c) and (d), obtain a basis of \( W_1 + W_2 \).

**SOLUTION.** Recall from Homework 3, Problem 3(b) that
\[
\text{span}(S_1) + \text{span}(S_2) = \text{span}(S_1 \cup S_2).
\]
So from (c) and (d),
\[
W_1 + W_2 = \text{span}\{[-1, 2, 1, 2]^\top, [1, 2, 3, 6]^\top\} + \text{span}\{[-1, 2, 1, 2]^\top, [1, -1, 1, 1]^\top\}
\]
\[
= \text{span}\{[-1, 2, 1, 2]^\top, [1, 2, 3, 6]^\top, [1, -1, 1, 1]^\top\}.
\]
Since
\[
\lambda_1 \begin{bmatrix}
-1 \\
2 \\
1 \\
2
\end{bmatrix} + \lambda_2 \begin{bmatrix}
1 \\
3 \\
-1 \\
1
\end{bmatrix} + \lambda_3 \begin{bmatrix}
1 \\
1 \\
1 \\
1
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]
implies \( \lambda_1 = \lambda_2 = \lambda_3 = 0 \). The vectors \( [-1, 2, 1, 2]^\top, [1, 2, 3, 6]^\top, [1, -1, 1, 1]^\top \} \) are linearly independent. Hence \( \{-1, 2, 1, 2\}^\top, [1, 2, 3, 6]^\top, [1, -1, 1, 1]^\top \} \) is a basis for \( W_1 + W_2 \).

4. Let \( W_1, W_2, W_3 \) be subspaces of a vector space \( V \).

(a) Show that
\[
\dim(W_1 + W_2) + \dim(W_1 \cap W_2) = \dim(W_1) + \dim(W_2).
\]

**SOLUTION.** Let \( \dim(W_1 \cap W_2) = k \) and let \( \{u_1, \ldots, u_k\} \) be a basis for \( W_1 \cap W_2 \). Note that \( \{u_1, \ldots, u_k, v_1, \ldots, v_m\} \) is a linearly independent subset of \( W_1 \) and by the Adding-On Algorithm, we may add vectors \( v_1, \ldots, v_m \) to it so that
\[
\{u_1, \ldots, u_k, v_1, \ldots, v_m\}
\]
is a basis for \( W_1 \). Applying the same argument now to \( W_2 \), we may also add vectors \( w_1, \ldots, w_n \) so that
\[
\{u_1, \ldots, u_k, w_1, \ldots, w_n\}
\]
is a basis for \( W_2 \). Hence
\[
\dim(W_1) = k + m, \quad \dim(W_2) = k + n.
\]
From Homework 3, Problem 3(b), we know that
\[
W_1 + W_2 = \text{span}\{\{u_1, \ldots, u_k, v_1, \ldots, v_m\} \cup \{u_1, \ldots, u_k, w_1, \ldots, w_n\}\}
\]
\[
= \text{span}\{u_1, \ldots, u_k, v_1, \ldots, v_m, w_1, \ldots, w_n\}.
\]
We will show that \( \{u_1, \ldots, u_k, v_1, \ldots, v_m, w_1, \ldots, w_n\} \) is linearly independent. Suppose
\[
\alpha_1 u_1 + \cdots + \alpha_k u_k + \beta_1 v_1 + \cdots + \beta_m v_m + \gamma_1 w_1 + \cdots + \gamma_n w_n = 0. \tag{4.3}
\]
We let
\[
v = \alpha_1 u_1 + \cdots + \alpha_k u_k + \beta_1 v_1 + \cdots + \beta_m v_m \in W_1. \tag{4.4}
\]
Then (4.3) implies that
\[
v = -\gamma_1 w_1 - \cdots - \gamma_n w_n \in W_2. \tag{4.5}
\]
So \( v \in W_1 \cap W_2 \) and since this has \( \{u_1, \ldots, u_k\} \) as a basis, there exist \( \lambda_1, \ldots, \lambda_k \) such that
\[
v = \lambda_1 u_1 + \cdots + \lambda_k u_k. \tag{4.6}
\]
Now (4.5) and (4.6) together implies that
\[ \lambda_1 u_1 + \cdots + \lambda_k u_k + \gamma_1 w_1 + \cdots + \gamma_n w_n = 0; \]
and since \{u_1, \ldots, u_k, w_1, \ldots, w_n\} is linearly independent, we get
\[ \lambda_1 = \cdots = \lambda_k = \gamma_1 = \cdots = \gamma_n = 0. \]
Substituting \(\gamma_1 = \cdots = \gamma_n = 0\) into (4.3) gives
\[ \alpha_1 u_1 + \cdots + \alpha_k u_k + \beta_1 v_1 + \cdots + \beta_m v_m = 0; \]
and since \{u_1, \ldots, u_k, v_1, \ldots, v_n\} is linearly independent, we get
\[ \alpha_1 = \cdots = \alpha_k = \beta_1 = \cdots = \beta_m = 0. \]
Hence this shows that \{u_1, \ldots, u_k, v_1, \ldots, v_m, w_1, \ldots, w_n\} is linearly independent and is thus a basis for \(W_1 + W_2\). Therefore \(\dim(W_1 + W_2) = k + m + n\). Finally, we have
\[ \dim(W_1 + W_2) + \dim(W_1 \cap W_2) = (k + m + n) + k \]
\[ = (k + m) + (k + n) \]
\[ = \dim(W_1) + \dim(W_2). \]

(b) Suppose \(\dim(W_1 + W_2) = \dim(W_1 \cap W_2) + 1\). Show that either \(W_1 \subseteq W_2\) or \(W_2 \subseteq W_1\).

**SOLUTION.** Since
\[ W_1 \cap W_2 \subseteq W_1 \subseteq W_1 + W_2, \]
by Theorem 3.14,
\[ \dim(W_1 \cap W_2) \leq \dim(W_1) \leq \dim(W_1 + W_2). \]

From the condition given, we must have either
\[ \dim(W_1) = \dim(W_1 \cap W_2) \]
and in which case
\[ W_1 = W_1 \cap W_2, \] (4.7)

or we have
\[ \dim(W_1) = \dim(W_1 + W_2) \]
and in which case
\[ W_1 = W_1 + W_2. \] (4.8)

By Problems 2(b) and 2(c) in Homework 2, the case (4.7) implies that \(W_1 \subseteq W_2\) while the case (4.8) implies \(W_2 \subseteq W_1\).

(c) Show that
\[ \dim(W_1 \cap W_2 \cap W_3) \geq \dim(W_1) + \dim(W_2) + \dim(W_3) - 2\dim(V). \]

**SOLUTION.** This follows directly from applying the formula in (a) twice:
\[ \dim(W_1 \cap W_2 \cap W_3) = \dim(W_1) + \dim(W_2 \cap W_3) - \dim(W_1 + (W_2 \cap W_3)) \]
\[ = \dim(W_1) + \dim(W_2) + \dim(W_3) - \dim(W_2 + W_3) - \dim(W_1 + (W_2 \cap W_3)) \]
\[ \geq \dim(W_1) + \dim(W_2) + \dim(W_3) - 2\dim(V) \]
where the last inequality follows from \(\dim(W_2 + W_3) \leq \dim(V)\) and \(\dim(W_1 + (W_2 \cap W_3)) \leq \dim(V)\) (by Theorem 3.14).
5. Let $V$ be a vector space over $\mathbb{R}$. We have seen in Homework 1 Problem 2 that $W = V \times V$ may be made into a vector space over $\mathbb{C}$ with appropriate addition and scalar multiplication. $W$ is called the complexification of $V$. Show that $\dim_{\mathbb{C}}(W) = \dim_{\mathbb{R}}(V)$.

SOLUTION. Let $\dim_{\mathbb{R}}(V) = n$ and $B = \{e_1, e_2, \ldots, e_n\}$ be a basis of $V$. We claim that

$B' = \{(e_1, 0), (e_2, 0), \ldots, (e_n, 0)\}$

is a basis of $W$ and so $\dim_{\mathbb{C}}(W) = n$ too. Let $(u, v) \in W$. Then since $u \in V$ and $v \in V$, there exists $a_1, \ldots, a_n \in \mathbb{R}$ and $b_1, \ldots, b_n \in \mathbb{R}$ such that

$u = a_1 e_1 + \cdots + a_n e_n$ and $v = b_1 e_1 + \cdots + b_n e_n$.

Hence

$(u, v) = (a_1 e_1 + \cdots + a_n e_n, b_1 e_1 + \cdots + b_n e_n)$

$= (a_1 e_1, b_1 e_1) \oplus \cdots \oplus (a_n e_n, b_n e_n)$

$= (a_1 + b_1 i) \lhd (e_1, 0) \oplus \cdots \oplus (a_n + b_n i) \lhd (e_n, 0)$

since

$(a_j + b_j i) \lhd (e_j, 0) = (a_j e_j - b_j 0, b_j e_j + a_j 0) = (a_j e_j, b_j e_j)$.

Hence $B'$ spans $W$. To show linear independence, let $a_1 + b_1 i, \ldots, a_n + b_n i \in \mathbb{C}$ be such that

$(a_1 + b_1 i) \lhd (e_1, 0) \oplus \cdots \oplus (a_n + b_n i) \lhd (e_n, 0) = (0, 0)$,

that is,

$(a_1 e_1 + \cdots + a_n e_n, b_1 e_1 + \cdots + b_n e_n) = (0, 0)$,

which is equivalent to

$a_1 e_1 + \cdots + a_n e_n = 0$ and $b_1 e_1 + \cdots + b_n e_n = 0$.

So $a_1 = \cdots = a_n = 0$ and $b_1 = \cdots = b_n = 0$ by the linear independence of $e_1, \ldots, e_n$ and so

$a_1 + b_1 i = \cdots = a_n + b_n i = 0 + 0 i$.

Hence $(e_1, 0), \ldots, (e_n, 0)$ are linearly independent over $\mathbb{C}$.