MATH 110: LINEAR ALGEBRA
SPRING 2007/08
PROBLEM SET 4 SOLUTIONS

1. Let W_1, W_2 be subspaces of V such that $V = W_1 \oplus W_2$. Let W be a subspace of V. Show that if $W_1 \subseteq W$ or $W_2 \subseteq W$, then

$$W = (W \cap W_1) \oplus (W \cap W_2).$$

Is this still true if we omit the condition '$W_1 \subseteq W$ or $W_2 \subseteq W$'?

Solution. Without loss of generality, we will assume that $W_1 \subseteq W$. So $W \cap W_1 = W_1$ and we just need to show that

$$W = W_1 \oplus (W \cap W_2).$$

By Problem **2**(b) in Homework **2**, the smallest subspace containing both W_1 and $W \cap W_2$ is $W_1 + (W \cap W_2)$; since $W_1 \subseteq W$ and $W \cap W_2 \subseteq W$, it follows that

$$W_1 + (W \cap W_2) \subseteq W. \tag{1.1}$$

To show the reverse inclusion, let $\mathbf{w} \in W$. Then $\mathbf{w} \in V$. Since $V = W_1 \oplus W_2$, there exist $\mathbf{w}_1 \in W_1$ and $\mathbf{w}_2 \in W_2$ such that

$$\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2.$$

Observe that

$$\mathbf{w}_2 = \mathbf{w} - \mathbf{w}_1 \in W \cap W_2$$

since $\mathbf{w}_2 \in W_2$ and $\mathbf{w} - \mathbf{w}_1 \in W$. Hence $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2 \in W_1 + (W \cap W_2)$. Since this is true for arbitrary $\mathbf{w} \in W$, we conclude that

$$W \subseteq W_1 + (W \cap W_2). \tag{1.2}$$

By (1.1) and (1.2),

$$W = W_1 + (W \cap W_2).$$

This is a direct sum since

$$(W \cap W_1) \cap (W \cap W_2) \subseteq W_1 \cap W_2 = \{\mathbf{0}\}.$$

The statement is false without the condition '$W_1 \subseteq W$ or $W_2 \subseteq W$'. Here is a counter-example. Let $W = \{(x, y) \in \mathbb{R}^2 \mid x = y\}$, $W_1 = \{(x, y) \in \mathbb{R}^2 \mid y = 0\}$, and $W_2 = \{(x, y) \in \mathbb{R}^2 \mid x = 0\}$. Then $W_1 \oplus W_2 = \mathbb{R}^2$ since every $(x, y) \in \mathbb{R}^2$ may be written uniquely as $(x, y) = (x, 0) + (0, y)$. However, $W \cap W_1 = \{(0, 0)\} = W \cap W_2$ and so

$$(W \cap W_1) \oplus (W \cap W_2) = \{(0, 0)\} \neq W.$$

2. For the following vector spaces V, find the coordinate representation of the respective elements.

(a) $V = \mathbb{P}_2 = \{ax^2 + bx + c \mid a, b, c \in \mathbb{R}\}$. Find $[p(x)]_{\mathscr{B}}$ where

$$p(x) = 2x^2 - 5x + 6, \quad \mathscr{B} = [1, x - 1, (x - 1)^2].$$

Solution. Since

$$p(x) = 2x^2 - 5x + 6 = 3 - (x - 1) + 2(x - 1)^2,$$

so

$$[p(x)]_{\mathscr{B}} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}.$$

Date: March 15, 2008 (Version 1.1).

(b) $V = \mathbb{R}^{2\times 2}$. Find $[A]_{\mathscr{B}}$ where

$$A = \begin{bmatrix} 2 & 3 \\ 4 & -7 \end{bmatrix}, \quad \mathscr{B} = \left[\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right].$$

Solution. Let

$$\begin{bmatrix} 2 & 3 \\ 4 & -7 \end{bmatrix} = \alpha \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \beta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \gamma \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + \delta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then solving

$$\begin{aligned} \alpha + \gamma + \delta &= 2, \\ \alpha - \beta - \gamma &= 3, \\ \alpha + \beta &= 4, \\ \alpha &= -7, \end{aligned}$$

gives $\alpha = -7, \beta = 11, \beta = -21, \delta = 30$. So

$$[A]_{\mathscr{B}} = \begin{bmatrix} -7 \\ 11 \\ -21 \\ 30 \end{bmatrix}.$$

(c) $V = \mathbb{R}^2$. Let $\theta \in \mathbb{R}$ be fixed. Find $[\mathbf{v}]_{\mathscr{B}}$ where

$$\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \mathscr{B} = \left[\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \right].$$

Solution. Let

$$\begin{bmatrix} x \\ y \end{bmatrix} = \alpha \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + \beta \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}.$$

Then solving

$$\begin{aligned} x &= \alpha\cos\theta - \beta\sin\theta, \\ y &= \alpha\sin\theta + \beta\cos\theta, \end{aligned}$$

gives

$$\begin{aligned} \alpha &= x\cos\theta + y\sin\theta, \\ \beta &= -x\sin\theta + y\cos\theta. \end{aligned}$$

So

$$[\mathbf{v}]_{\mathscr{B}} = \begin{bmatrix} x\cos\theta + y\sin\theta \\ -x\sin\theta + y\cos\theta \end{bmatrix}.$$

3. Let W_1 and W_2 be the following subspaces of $\mathbb{R}^4$.

$$W_1 = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 6 \\ 12 \end{bmatrix} \right\}, \quad W_2 = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \\ 5 \end{bmatrix} \right\}.$$

(a) Find a basis of $W_1 \cap W_2$.
Solution. Let the three vectors spanning W_1 be $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ and the two vectors spanning W_2 be $\mathbf{v}_1, \mathbf{v}_2$ respectively. If $\mathbf{w} \in W_1 \cap W_2$, then there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ and $\beta_1, \beta_2 \in \mathbb{R}$ such that

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 = \mathbf{w} = \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2.$$

In other words, we want to solve

$$\alpha_1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 6 \end{bmatrix} + \alpha_2 \begin{bmatrix} 4 \\ -1 \\ 3 \\ 6 \end{bmatrix} + \alpha_3 \begin{bmatrix} 5 \\ 1 \\ 6 \\ 12 \end{bmatrix} = \beta_1 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} + \beta_2 \begin{bmatrix} 2 \\ -1 \\ 4 \\ 5 \end{bmatrix}.$$

Forming the augmented system to solve for $\alpha_1, \alpha_2, \alpha_3$ in terms of β_1, β_2 (alternatively, we could also solve for β_1 and β_2 in terms of $\alpha_1, \alpha_2, \alpha_3$), we get upon Gauss-Jordan elimination,

$$\left[\begin{array}{ccc|c} 1 & 4 & 5 & \beta_1 + 2\beta_2 \\ 2 & -1 & 1 & -\beta_1 - \beta_2 \\ 3 & 3 & 6 & \beta_1 + 4\beta_2 \\ 6 & 6 & 12 & \beta_1 + 5\beta_2 \end{array}\right] \longrightarrow \cdots \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & -\frac{1}{4}\beta_1 - \frac{2}{9}\beta_2 \\ 0 & 1 & 1 & \frac{1}{3}\beta_1 + \frac{5}{9}\beta_2 \\ 0 & 0 & 0 & \beta_1 + 3\beta_2 \\ 0 & 0 & 0 & -\beta_1 - 3\beta_2 \end{array}\right].$$

For this system to be consistent, we must have $\beta_1 + 3\beta_2 = 0$. Hence

$$\mathbf{w} = \beta_1 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} + \beta_2 \begin{bmatrix} 2 \\ -1 \\ 4 \\ 5 \end{bmatrix} = \beta_2 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

for some $\beta_2 \in \mathbb{R}$. In other words,

$$W_1 \cap W_2 = \operatorname{span}\{[-1, 2, 1, 2]^\top\}.$$

(b) Find a basis of $W_1 + W_2$.
SOLUTION. We will do this in (e). For an alternative way, see the solution to Homework **5**, Problem **1** from Fall 2007.

(c) Extend the basis of $W_1 \cap W_2$ in (a) to get a basis of W_1.
SOLUTION. We will apply the Adding-On Algorithm to $\{[-1, 2, -1, 2]^\top\}$ and check each of the three vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ in turn. Since

$$\begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \stackrel{?}{=} \lambda \begin{bmatrix} -1 \\ 2 \\ -1 \\ 2 \end{bmatrix}$$

clearly has no solution, we add $\mathbf{u}_1$ to get $\{[-1, 2, 1, 2]^\top, [1, -1, 1, 1]^\top\}$. Since

$$\begin{bmatrix} 4 \\ -1 \\ 3 \\ 6 \end{bmatrix} = \lambda_1 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 2 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 6 \end{bmatrix}$$

has a solution $\lambda_1 = -9/4$, $\lambda_2 = 7/4$, $\mathbf{u}_2$ need not be added. Since

$$\begin{bmatrix} 5 \\ 1 \\ 6 \\ 12 \end{bmatrix} = \lambda_1 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 2 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 6 \end{bmatrix}$$

has a solution $\lambda_1 = -9/4$, $\lambda_2 = 11/4$, $\mathbf{u}_3$ need not be added. Hence $\{[-1, 2, 1, 2]^\top, [1, 2, 3, 6]^\top\}$ is a basis for W_1 by the Adding-On Algorithm.

(d) Extend the basis of $W_1 \cap W_2$ in (a) to get a basis of W_2.
SOLUTION. We will apply the Adding-On Algorithm to $\{[-1, 2, -1, 2]^\top\}$ and check each of the two vectors $\mathbf{v}_1, \mathbf{v}_2$ in turn. Since

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 6 \end{bmatrix} \stackrel{?}{=} \lambda \begin{bmatrix} -1 \\ 2 \\ -1 \\ 2 \end{bmatrix}$$

clearly has no solution, we add $\mathbf{v}_1$ to get $\{[-1,2,-1,2]^\top, [1,2,3,6]^\top\}$. Since

$$\begin{bmatrix} 2 \\ -1 \\ 4 \\ 5 \end{bmatrix} = \lambda_1 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 2 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

has a solution $\lambda_1 = 1$, $\lambda_2 = 3$, $\mathbf{v}_2$ need not be added. Hence $\{[-1,2,1,2]^\top, [1,-1,1,1]^\top\}$ is a basis for W_2 by the Adding-On Algorithm.

(e) From the bases in (c) and (d), obtain a basis of $W_1 + W_2$.

SOLUTION. Recall from Homework **3**, Problem **3**(b) that

$$\operatorname{span}(S_1) + \operatorname{span}(S_2) = \operatorname{span}(S_1 \cup S_2).$$

So from (c) and (d),

$$\begin{aligned} W_1 + W_2 &= \operatorname{span}\{[-1,2,1,2]^\top, [1,2,3,6]^\top\} + \operatorname{span}\{[-1,2,1,2]^\top, [1,-1,1,1]^\top\} \\ &= \operatorname{span}\{[-1,2,1,2]^\top, [1,2,3,6]^\top, [1,-1,1,1]^\top\}. \end{aligned}$$

Since

$$\lambda_1 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 2 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 6 \end{bmatrix} + \lambda_3 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

implies $\lambda_1 = \lambda_2 = \lambda_3 = 0$. The vectors $[-1,2,1,2]^\top, [1,2,3,6]^\top, [1,-1,1,1]^\top$ are linearly independent. Hence $\{[-1,2,1,2]^\top, [1,2,3,6]^\top, [1,-1,1,1]^\top\}$ is a basis for $W_1 + W_2$.

4. Let W_1, W_2, W_3 be subspaces of a vector space V.

(a) Show that

$$\dim(W_1 + W_2) + \dim(W_1 \cap W_2) = \dim(W_1) + \dim(W_2).$$

SOLUTION. Let $\dim(W_1 \cap W_2) = k$ and let $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ be a basis for $W_1 \cap W_2$. Note that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a linearly independent subset of W_1 and by the Adding-On Algorithm, we may add vectors $\mathbf{v}_1, \dots, \mathbf{v}_m$ to it so that

$$\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{v}_1, \dots, \mathbf{v}_m\}$$

is a basis for W_1. Applying the same argument now to W_2, we may also add vectors $\mathbf{w}_1, \dots, \mathbf{w}_n$ so that

$$\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{w}_1, \dots, \mathbf{w}_n\}$$

is a basis for W_2. Hence

$$\dim(W_1) = k + m, \quad \dim(W_2) = k + n.$$

From Homework **3**, Problem **3**(b), we know that

$$\begin{aligned} W_1 + W_2 &= \operatorname{span}(\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{v}_1, \dots, \mathbf{v}_m\} \cup \{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{w}_1, \dots, \mathbf{w}_n\}) \\ &= \operatorname{span}\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{v}_1, \dots, \mathbf{v}_m, \mathbf{w}_1, \dots, \mathbf{w}_n\}. \end{aligned}$$

We will show that $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{v}_1, \dots, \mathbf{v}_m, \mathbf{w}_1, \dots, \mathbf{w}_n\}$ is linearly independent. Suppose

$$\alpha_1 \mathbf{u}_1 + \cdots + \alpha_k \mathbf{u}_k + \beta_1 \mathbf{v}_1 + \cdots + \beta_m \mathbf{v}_m + \gamma_1 \mathbf{w}_1 + \cdots + \gamma_n \mathbf{w}_n = \mathbf{0}. \tag{4.3}$$

We let

$$\mathbf{v} = \alpha_1 \mathbf{u}_1 + \cdots + \alpha_k \mathbf{u}_k + \beta_1 \mathbf{v}_1 + \cdots + \beta_m \mathbf{v}_m \in W_1. \tag{4.4}$$

Then (4.3) implies that

$$\mathbf{v} = -\gamma_1 \mathbf{w}_1 - \cdots - \gamma_n \mathbf{w}_n \in W_2. \tag{4.5}$$

So $\mathbf{v} \in W_1 \cap W_2$ and since this has $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ as a basis, there exist $\lambda_1, \dots, \lambda_k$ such that

$$\mathbf{v} = \lambda_1 \mathbf{u}_1 + \cdots + \lambda_k \mathbf{u}_k. \tag{4.6}$$

Now (4.5) and (4.6) together implies that

$$\lambda_1\mathbf{u}_1 + \cdots + \lambda_k\mathbf{u}_k + \gamma_1\mathbf{w}_1 + \cdots + \gamma_n\mathbf{w}_n = \mathbf{0};$$

and since $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{w}_1, \dots, \mathbf{w}_n\}$ is linearly independent, we get

$$\lambda_1 = \cdots = \lambda_k = \gamma_1 = \cdots = \gamma_n = 0.$$

Substituting $\gamma_1 = \cdots = \gamma_n = 0$ into (4.3) gives

$$\alpha_1\mathbf{u}_1 + \cdots + \alpha_k\mathbf{u}_k + \beta_1\mathbf{v}_1 + \cdots + \beta_m\mathbf{v}_m = \mathbf{0};$$

and since $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{v}_1, \dots, \mathbf{v}_n\}$ is linearly independent, we get

$$\alpha_1 = \cdots = \alpha_k = \beta_1 = \cdots = \beta_m = 0.$$

Hence this shows that $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{v}_1, \dots, \mathbf{v}_m, \mathbf{w}_1, \dots, \mathbf{w}_n\}$ is linearly independent and is thus a basis for $W_1 + W_2$. Therefore $\dim(W_1 + W_2) = k + m + n$. Finally, we have

$$\begin{aligned}\dim(W_1 + W_2) + \dim(W_1 \cap W_2) &= (k + m + n) + k \\ &= (k + m) + (k + n) \\ &= \dim(W_1) + \dim(W_2).\end{aligned}$$

(b) Suppose $\dim(W_1 + W_2) = \dim(W_1 \cap W_2) + 1$. Show that either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.
Solution. Since

$$W_1 \cap W_2 \subseteq W_1 \subseteq W_1 + W_2,$$

by Theorem **3.14**,

$$\dim(W_1 \cap W_2) \leq \dim(W_1) \leq \dim(W_1 + W_2).$$

From the condition given, we must have either

$$\dim(W_1) = \dim(W_1 \cap W_2)$$

and in which case

$$W_1 = W_1 \cap W_2, \tag{4.7}$$

or we have

$$\dim(W_1) = \dim(W_1 + W_2)$$

and in which case

$$W_1 = W_1 + W_2. \tag{4.8}$$

By Problems **2**(b) and **2**(c) in Homework **2**, the case (4.7) implies that $W_1 \subseteq W_2$ while the case (4.8) implies $W_2 \subseteq W_1$.

(c) Show that

$$\dim(W_1 \cap W_2 \cap W_3) \geq \dim(W_1) + \dim(W_2) + \dim(W_3) - 2\dim(V).$$

Solution. This follows directly from applying the formula in (a) twice:

$$\begin{aligned}\dim(W_1 \cap W_2 \cap W_3) &= \dim(W_1) + \dim(W_2 \cap W_3) - \dim(W_1 + (W_2 \cap W_3)) \\ &= \dim(W_1) + \dim(W_2) + \dim(W_3) - \dim(W_2 + W_3) - \dim(W_1 + (W_2 \cap W_3)) \\ &\geq \dim(W_1) + \dim(W_2) + \dim(W_3) - 2\dim(V)\end{aligned}$$

where the last inequality follows from $\dim(W_2 + W_3) \leq \dim(V)$ and $\dim(W_1 + (W_2 \cap W_3)) \leq \dim(V)$ (by Theorem **3.14**).

5. Let V be a vector space over $\mathbb{R}$. We have seen in Homework **1** Problem **2** that $W = V \times V$ may be made into a vector space over $\mathbb{C}$ with appropriate addition and scalar multiplication. W is called the *complexification* of V. Show that

$$\dim_{\mathbb{C}}(W) = \dim_{\mathbb{R}}(V).$$

SOLUTION. Let $\dim_{\mathbb{R}}(V) = n$ and $\mathscr{B} = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ be a basis of V. We claim that

$$\mathscr{B}' = \{(\mathbf{e}_1, \mathbf{0}), (\mathbf{e}_2, \mathbf{0}), \dots, (\mathbf{e}_n, \mathbf{0})\}$$

is a basis of W and so $\dim_{\mathbb{C}}(W) = n$ too. Let $(\mathbf{u}, \mathbf{v}) \in W$. Then since $\mathbf{u} \in V$ and $\mathbf{v} \in V$, there exists $a_1, \dots, a_n \in \mathbb{R}$ and $b_1, \dots, b_n \in \mathbb{R}$ such that

$$\mathbf{u} = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n \quad \text{and} \quad \mathbf{v} = b_1\mathbf{e}_1 + \cdots + b_n\mathbf{e}_n.$$

Hence

$$\begin{aligned}(\mathbf{u}, \mathbf{v}) &= (a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n, b_1\mathbf{e}_1 + \cdots + b_n\mathbf{e}_n)\\ &= (a_1\mathbf{e}_1, b_1\mathbf{e}_1) \boxplus \cdots \boxplus (a_n\mathbf{e}_n, b_n\mathbf{e}_n)\\ &= (a_1 + b_1 i) \boxdot (\mathbf{e}_1, \mathbf{0}) \boxplus \cdots \boxplus (a_n + b_n i) \boxdot (\mathbf{e}_n, \mathbf{0})\end{aligned}$$

since

$$(a_j + b_j i) \boxdot (\mathbf{e}_j, \mathbf{0}) = (a_j\mathbf{e}_j - b_j\mathbf{0}, b_j\mathbf{e}_j + a_j\mathbf{0}) = (a_j\mathbf{e}_j, b_j\mathbf{e}_j).$$

Hence $\mathscr{B}'$ spans W. To show linear independence, let $a_1 + b_1 i, \dots, a_n + b_n i \in \mathbb{C}$ be such that

$$(a_1 + b_1 i) \boxdot (\mathbf{e}_1, \mathbf{0}) \boxplus \cdots \boxplus (a_n + b_n i) \boxdot (\mathbf{e}_n, \mathbf{0}) = (\mathbf{0}, \mathbf{0}),$$

that is,

$$(a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n, b_1\mathbf{e}_1 + \cdots + b_n\mathbf{e}_n) = (\mathbf{0}, \mathbf{0}),$$

which is equivalent to

$$a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n = \mathbf{0} \quad \text{and} \quad b_1\mathbf{e}_1 + \cdots + b_n\mathbf{e}_n = \mathbf{0}.$$

So $a_1 = \cdots = a_n = 0$ and $b_1 = \cdots = b_n = 0$ by the linear independence of $\mathbf{e}_1, \dots, \mathbf{e}_n$ and so

$$a_1 + b_1 i = \cdots = a_n + b_n i = 0 + 0i.$$

Hence $(\mathbf{e}_1, \mathbf{0}), \dots, (\mathbf{e}_n, \mathbf{0})$ are linearly independent over $\mathbb{C}$.