If $V$ is a vector space over $\mathbb{F}$, we will write $\dim_{\mathbb{F}}(V)$ for the dimension of $V$ when we wish to emphasize the field of scalars. For example, $\dim_{\mathbb{C}}(\mathbb{C}^3) = 3$ and $\dim_{\mathbb{R}}(\mathbb{C}^3) = 6$.

1. Let $W_1, W_2$ be subspaces of $V$ such that $V = W_1 \oplus W_2$. Let $W$ be a subspace of $V$. Show that if $W_1 \subseteq W$ or $W_2 \subseteq W$, then

$$W = (W \cap W_1) \oplus (W \cap W_2).$$

Is this still true if we omit the condition ‘$W_1 \subseteq W$ or $W_2 \subseteq W$’?

2. For the following vector spaces $V$, find the coordinate representation of the respective elements.

(a) $V = \mathbb{P}_2 = \{ ax^2 + bx + c \mid a, b, c \in \mathbb{R} \}$. Find $[p(x)]_{\mathcal{B}}$ where $p(x) = 2x^2 - 5x + 6$, $\mathcal{B} = [1, x - 1, (x - 1)^2]$.

(b) $V = \mathbb{R}^{2 \times 2}$. Find $[A]_{\mathcal{B}}$ where $A = \begin{bmatrix} 2 & 3 \\ 4 & -7 \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & -1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$.

(c) $V = \mathbb{R}^2$. Let $\theta \in \mathbb{R}$ be fixed. Find $[v]_{\mathcal{B}}$ where $v = \begin{bmatrix} x \\ y \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} \cos \theta \\ -\sin \theta \\ \sin \theta \\ \cos \theta \end{bmatrix}$.

3. Let $W_1$ and $W_2$ be the following subspaces of $\mathbb{R}^4$.

$$W_1 = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 6 \\ 12 \end{bmatrix} \right\}, \quad W_2 = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \\ 5 \end{bmatrix} \right\}.$$ 

(a) Find a basis of $W_1 \cap W_2$.
(b) Find a basis of $W_1 + W_2$.
(c) Extend the basis of $W_1 \cap W_2$ in (a) to get a basis of $W_1$.
(d) Extend the basis of $W_1 \cap W_2$ in (a) to get a basis of $W_2$.
(e) From the bases in (c) and (d), obtain a basis of $W_1 + W_2$.

4. Let $W_1, W_2, W_3$ be subspaces of a vector space $V$.

(a) Show that

$$\dim(W_1 + W_2) + \dim(W_1 \cap W_2) = \dim(W_1) + \dim(W_2).$$

(b) Suppose $\dim(W_1 + W_2) = \dim(W_1 \cap W_2) + 1$. Show that either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

(c) Show that

$$\dim(W_1 \cap W_2 \cap W_3) \geq \dim(W_1) + \dim(W_2) + \dim(W_3) - 2 \dim(V).$$
5. Let $V$ be a vector space over $\mathbb{R}$. We have seen in Homework 1 Problem 2 that $W = V \times V$ may be made into a vector space over $\mathbb{C}$ with appropriate addition and scalar multiplication. $W$ is called the *complexification* of $V$. Show that

$$\dim_\mathbb{C}(W) = \dim_\mathbb{R}(V).$$