MATH 110: LINEAR ALGEBRA
SPRING 2007/08
PROBLEM SET 4

If V is a vector space over $\mathbb{F}$, we will write $\dim_{\mathbb{F}}(V)$ for the dimension of V when we wish to emphasize the field of scalars. For example, $\dim_{\mathbb{C}}(\mathbb{C}^3) = 3$ and $\dim_{\mathbb{R}}(\mathbb{C}^3) = 6$.

1. Let W_1, W_2 be subspaces of V such that $V = W_1 \oplus W_2$. Let W be a subspace of V. Show that if $W_1 \subseteq W$ or $W_2 \subseteq W$, then

$$W = (W \cap W_1) \oplus (W \cap W_2).$$

Is this still true if we omit the condition '$W_1 \subseteq W$ or $W_2 \subseteq W$'?

2. For the following vector spaces V, find the coordinate representation of the respective elements.

(a) $V = \mathbb{P}_2 = \{ax^2 + bx + c \mid a, b, c \in \mathbb{R}\}$. Find $[p(x)]_{\mathscr{B}}$ where

$$p(x) = 2x^2 - 5x + 6, \quad \mathscr{B} = [1, x-1, (x-1)^2].$$

(b) $V = \mathbb{R}^{2\times 2}$. Find $[A]_{\mathscr{B}}$ where

$$A = \begin{bmatrix} 2 & 3 \\ 4 & -7 \end{bmatrix}, \quad \mathscr{B} = \left[\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right].$$

(c) $V = \mathbb{R}^2$. Let $\theta \in \mathbb{R}$ be fixed. Find $[\mathbf{v}]_{\mathscr{B}}$ where

$$\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \mathscr{B} = \left[\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \right].$$

3. Let W_1 and W_2 be the following subspaces of $\mathbb{R}^4$.

$$W_1 = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 6 \\ 12 \end{bmatrix} \right\}, \quad W_2 = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \\ 5 \end{bmatrix} \right\}.$$

(a) Find a basis of $W_1 \cap W_2$.
(b) Find a basis of $W_1 + W_2$.
(c) Extend the basis of $W_1 \cap W_2$ in (a) to get a basis of W_1.
(d) Extend the basis of $W_1 \cap W_2$ in (a) to get a basis of W_2.
(e) From the bases in (c) and (d), obtain a basis of $W_1 + W_2$.

4. Let W_1, W_2, W_3 be subspaces of a vector space V.

(a) Show that

$$\dim(W_1 + W_2) + \dim(W_1 \cap W_2) = \dim(W_1) + \dim(W_2).$$

(b) Suppose $\dim(W_1 + W_2) = \dim(W_1 \cap W_2) + 1$. Show that either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

(c) Show that

$$\dim(W_1 \cap W_2 \cap W_3) \geq \dim(W_1) + \dim(W_2) + \dim(W_3) - 2\dim(V).$$

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5. Let V be a vector space over $\mathbb{R}$. We have seen in Homework **1** Problem **2** that $W = V \times V$ may be made into a vector space over $\mathbb{C}$ with appropriate addition and scalar multiplication. W is called the *complexification* of V. Show that

$$\dim_{\mathbb{C}}(W) = \dim_{\mathbb{R}}(V).$$