1. Let $V$ be a vector space over $\mathbb{F}$. Let $w \in V$ be a fixed non-zero vector and $\mu \in \mathbb{F}$ be a fixed non-zero scalar.
   (a) Show that the function $f : \mathbb{F} \rightarrow V$ defined by $f(\lambda) = \lambda w$ is injective.
   (b) Show that the function $g : V \rightarrow V$ defined by $g(v) = \mu v$ is bijective.
   (c) Show that the function $h : V \rightarrow V$ defined by $h(v) = v + w$ is bijective.

2. Let $W_1$ and $W_2$ be subspaces of a vector space $V$. The sum of $W_1$ and $W_2$ is the subset of $V$ defined by
   \[ W_1 + W_2 = \{ w_1 + w_2 \in V \mid w_1 \in W_1, w_2 \in W_2 \}. \]
   (a) Prove that $W_1 + W_2$ is a subspace of $V$.
   (b) Prove that $W_1 + W_2$ is the smallest subspace of $V$ containing both $W_1$ and $W_2$.
   (c) Prove that $W_1 \cap W_2$ is the largest subspace of $V$ contained in both $W_1$ and $W_2$.

3. Let $W_1$ and $W_2$ be subspaces of a vector space $V$. Show that the following statements are equivalent.
   (i) $W_1 \cap W_2 = \{0\}$.
   (ii) If $w_1 \in W_1$ and $w_2 \in W_2$ are such that $w_1 + w_2 = 0$, then $w_1 = w_2 = 0$.
   (iii) If $w_1 + w_2 = w'_1 + w'_2$, where $w_1, w'_1 \in W_1$ and $w_2, w'_2 \in W_2$, then $w_1 = w'_1$ and $w_2 = w'_2$.
   If any one of these equivalent conditions holds, then $W_1 + W_2$ is written $W_1 \oplus W_2$ and is called the direct sum of $W_1$ and $W_2$.

4. (a) State and prove the analogue of the statements in Problem 2 for the direct sum of three or more subspaces.
   (b) Let $W_1, W_2, W_3$ be subspaces of a vector space $V$. Suppose
   \[ W_1 \cap W_2 = W_1 \cap W_3 = W_2 \cap W_3 = \{0\}. \]
   Must $W_1 + W_2 + W_3$ be a direct sum?

5. Prove or provide a counter example for the following.
   (a) Let
   \[
   V_1 := \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \in \mathbb{R}^{2 \times 2} \mid a, b \in \mathbb{R} \right\}, \\
   V_2 := \left\{ \begin{bmatrix} c & d \\ d & -c \end{bmatrix} \in \mathbb{R}^{2 \times 2} \mid c, d \in \mathbb{R} \right\}.
   
   Is it true that $\mathbb{R}^{2 \times 2} = V_1 \oplus V_2$?
   
   (b) Let
   \[
   W_1 := \{ p(x) \in \mathbb{P}_3 \mid p(-x) = p(x) \text{ for all } x \in \mathbb{R} \}, \\
   W_2 := \{ p(x) \in \mathbb{P}_3 \mid p(-x) = -p(x) \text{ for all } x \in \mathbb{R} \}.
   
   Is it true that $\mathbb{P}_3 = W_1 \oplus W_2$?

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