1. Prove that the following are vector spaces over \( \mathbb{R} \):
   (a) polynomials of degree not more than \( d \),
   \[
   \mathbb{P}_d = \{ a_0 + a_1 x + \cdots + a_d x^d \mid a_i \in \mathbb{R} \text{ for all } i \},
   \]
   (b) \( m \)-by-\( n \) matrices
   \[
   \mathbb{R}^{m \times n} = \{ [a_{ij}]_{i=1}^m_{j=1} \mid a_{ij} \in \mathbb{R} \text{ for all } i,j \}.
   \]
   The addition and scalar multiplication operations for polynomials and matrices are as defined in the lectures.
   **Solution.** Routine.

2. Let \( V \) be a vector space over \( \mathbb{R} \) with addition and scalar multiplication denoted by \( + \) and \( \cdot \), respectively. Let \( W = V \times V = \{ (v_1, v_2) \mid v_1, v_2 \in V \} \). Prove that \( W \) is a vector space over \( \mathbb{C} \) with addition defined by
   \[
   (u_1, u_2) \oplus (v_1, v_2) = (u_1 + v_1, u_2 + v_2)
   \]
   for all \( (u_1, u_2), (v_1, v_2) \in W \) and scalar multiplication defined by
   \[
   (a + bi) \odot (v_1, v_2) = (a \cdot v_1 - b \cdot v_2, b \cdot v_1 + a \cdot v_2)
   \]
   for all \( a + bi \in \mathbb{C} \) and \( (v_1, v_2) \in W \). Here \( i = \sqrt{-1} \) and \( a, b \in \mathbb{R} \).
   **Solution.** Routine.

3. Which of the following are subspaces of \( \mathbb{R}^2 \)? Justify your answers.
   (a) \( U_a = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 0, \ x, y \in \mathbb{R} \} \),
   (b) \( U_b = \{ (x, y) \in \mathbb{R}^2 \mid x^2 - y = 0, \ x, y \in \mathbb{R} \} \),
   (c) \( U_c = \{ (x, y) \in \mathbb{R}^2 \mid x^2 - y = 0, \ x, y \in \mathbb{R} \} \),
   (d) \( U_d = \{ (x, y) \in \mathbb{R}^2 \mid x - y = 0, \ x, y \in \mathbb{R} \} \),
   (e) \( U_e = \{ (x, y) \in \mathbb{R}^2 \mid x - y = 1, \ x, y \in \mathbb{R} \} \).

   If we replace \( \mathbb{R} \) by \( \mathbb{C} \) and \( \mathbb{R}^2 \) by \( \mathbb{C}^2 \) above, will any of your answers change?
   **Solution.** Note that \( U_a = \{ (0, 0) \} \) and so is a subspace. Let \( \alpha, \beta \in \mathbb{R} \). \( U_d \) is a subspace by Theorem 1.8: if \( x_1 - y_1 = 0 \) and \( x_2 - y_2 = 0 \), then \( (\alpha x_1 + \beta x_2) - (\alpha x_1 + \beta x_2) = \alpha (x_1 - y_1) + \beta (x_2 - y_2) = 0 \). \( (0, 0) \notin U_e \) and so it is not a subspace. Note that \( (1, 1), (-1, 1) \in U_b \) (resp. \( U_c \)) but \( (1, 1) + (-1, 1) = (0, 2) \notin U_b \) (resp. \( U_c \)) and so \( U_b \) (resp. \( U_c \)) is not a subspace. Over \( \mathbb{C} \), \( U_a \) is not a subspace since \( (i, 1), (-i, 1) \in U_a \) but \( (i, 1) + (-i, 1) = (0, 2) \notin U_a \).

4. Which of the following are subspaces of \( \mathbb{P}_3 \)? Justify your answer. Here \( \mathbb{P}_3 \) denotes the vector space of polynomials of degree not more than 3, ie.
   \[
   \mathbb{P}_3 = \{ p(x) = a + bx + cx^2 + dx^3 \mid a, b, c, d \in \mathbb{R} \}.
   \]
   (a) \( V_a = \{ p(x) \in \mathbb{P}_3 \mid \text{degree of } p(x) \text{ is } 2 \} \),
   (b) \( V_b = \{ p(x) \in \mathbb{P}_3 \mid p(0) = p(1) \} \),
   (c) \( V_c = \{ p(x) \in \mathbb{P}_3 \mid p(0) = 1 \} \),
   (d) \( V_d = \{ p(x) \in \mathbb{P}_3 \mid p(1) = 0 \} \),
   (e) \( V_e = \{ p(x) \in \mathbb{P}_3 \mid p(x) \geq 0 \text{ for all } x \text{ with } -1 \leq x \leq 1 \} \),
   (f) \( V_f = \{ p(x) \in \mathbb{P}_3 \mid p(-x) = -p(x) \text{ for all } x \} \).

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5. Which of the following are subspaces of $\mathbb{R}^{2\times 2}$? Justify your answer. Here $\mathbb{R}^{2\times 2}$ denotes the vector space of $2 \times 2$ matrices, i.e.,

$$\mathbb{R}^{2\times 2} = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{R} \right\}.$$ 

(a) $W_a = \{ A \in \mathbb{R}^{2\times 2} \mid A^2 = A \}$, 
SOLUTION. Not a subspace. $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in W_a$ but $2I = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \notin W_a$.

(b) $W_b = \{ A \in \mathbb{R}^{2\times 2} \mid AB = BA \}$ where $B \in \mathbb{R}^{2\times 2}$ is a fixed matrix,
SOLUTION. Clearly $I \in W_b$ and so $W_b \neq \emptyset$. Let $A_1, A_2 \in W_b$ and $\lambda, \mu \in \mathbb{R}$. Since $A_1B = BA_1$ and $A_2B = BA_2$, we have

$$\begin{align*}
(\lambda A_1 + \mu A_2)B &= \lambda A_1 B + \mu A_2 B \\
&= \lambda BA_1 + \mu BA_2 \\
&= B(\lambda A_1 + \mu A_2),
\end{align*}$$

and so $\lambda A_1 + \mu A_2 \in W_b$. Hence $W_b$ is a subspace by Theorem 1.8.

(c) $W_c = \{ A \in \mathbb{R}^{2\times 2} \mid \det(A) = \alpha \}$ where $\alpha \in \mathbb{R}$ is a fixed scalar,
SOLUTION. For $\alpha \neq 0$, let $A \in W_c$. Observe that $\det(2A) = 4\det(A) = 4\alpha \neq \alpha$, and so $2A \notin W_c$. So $W_c$ is not a subspace for $\alpha \neq 0$. For $\alpha = 0$, consider $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
Since $A_1 + A_2 = I$ and $\det(I) = 1 \neq 0$, $A_1 + A_2 \notin W_c$. So $W_c$ is not a subspace for $\alpha = 0$ either.

(d) $W_d = \{ A \in \mathbb{R}^{2\times 2} \mid \text{tr}(A) = \alpha \}$ where $\alpha \in \mathbb{R}$ is a fixed scalar,
SOLUTION. For $\alpha \neq 0$, let $A \in W_d$. Observe that $\text{tr}(2A) = 2\text{tr}(A) = 2\alpha \neq \alpha$, and so $2A \notin W_d$. So $W_d$ is not a subspace for $\alpha \neq 0$. For $\alpha = 0$, let $A_1, A_2 \in W_d$ and $\lambda, \mu \in \mathbb{R}$. Since $\text{tr}(A_1) = 0 = \text{tr}(A_2)$, we have

$$\text{tr}(\lambda A_1 + \mu A_2) = \lambda \text{tr}(A_1) + \mu \text{tr}(A_2) = \lambda \cdot 0 + \mu \cdot 0 = 0,$$

and so $\lambda A_1 + \mu A_2 \in W_d$. Hence $W_d$ is a subspace by Theorem 1.8.

(e) $W_e = \{ A \in \mathbb{R}^{2\times 2} \mid Ax = b \}$ where $x, b \in \mathbb{R}^2$ are two fixed vectors,
SOLUTION. For $b \neq 0$, let $A \in W_e$. Observe that $2Ax = 2b \neq b$, and so $2A \notin W_e$. So $W_e$ is not a subspace for $b \neq 0$. For $b = 0$, let $A_1, A_2 \in W_e$ and $\lambda, \mu \in \mathbb{R}$. Since $A_1x = 0 = A_2x$, we have

$$\begin{align*}
(\lambda A_1 + \mu A_2)x &= \lambda A_1 x + \mu A_2 x = \lambda 0 + \mu 0 = 0,
\end{align*}$$

and so $\lambda A_1 + \mu A_2 \in W_e$. Hence $W_e$ is a subspace by Theorem 1.8.

(f) $W_f = \{ A \in \mathbb{R}^{2\times 2} \mid Ax = \lambda x \text{ for some } \lambda \in \mathbb{R} \}$ where $x \in \mathbb{R}^2$ is a fixed vector.
SOLUTION. Let $A_1, A_2 \in W_f$ and $\alpha_1, \alpha_2 \in \mathbb{R}$. Then there exists $\lambda_1$ and $\lambda_2$ such that $A_1x = \lambda_1x$ and $A_2x = \lambda_2x$.
So

$$\begin{align*}
(\alpha_1 A_1 + \alpha_2 A_2)x &= \alpha_1 A_1 x + \alpha_2 A_2 x \\
&= \alpha_1 \lambda_1 x + \alpha_2 \lambda_2 x \\
&= (\alpha_1 \lambda_1 + \alpha_2 \lambda_2)x,
\end{align*}$$

and so $\alpha_1 A_1 + \alpha_2 A_2 \in W_f$. Hence $W_f$ is a subspace by Theorem 1.8.

6. Let $V$ be a vector space over a field $F$ and $W$ be a subspace of $V$. 

(a) Let $0_V$ be the additive identity of $V$ and $0_W$ be the additive identity of $W$. Prove that $0_V = 0_W$.

**Solution.** By definition, $0_V + v = v = v + 0_V$ for all $v \in V$. Since $W \subseteq V$, we have that

$$0_V + w = w = w + 0_V$$

for all $w \in W$. In other words, $0_V$ is an additive identity of $W$. By the uniqueness of additive identity (Theorem 1.1) applied to the vector space $W$, it follows that $0_V = 0_W$.

(b) Let $w \in W$. So $w \in V$ in particular. Let $v \in V$ be the additive inverse of $w$ as an element of $V$. Let $v' \in W$ be the additive inverse of $w$ as an element of $W$. Prove that $v = v'$.

**Solution.** By our choice of $v$ and $v'$, $v + w = 0_V$ and $v' + w = 0_W$. Since $0_V = 0_W$, we have that

$$v + w = v' + w.$$ 

Adding $v$ to both sides of the equation and using additive associativity yields

$$v + (w + v) = v' + (w + v),$$

so

$$v + 0_V = v' + 0_V,$$

and so

$$v = v',$$

as required.