MATH 110: LINEAR ALGEBRA
FALL 2007/08
PROBLEM SET 6 SOLUTIONS

You may use without proof any results from the lectures as well as any basic properties of matrix operations and calculus operations. $\mathbb{P}$ will denote the vector space of polynomials in x with coefficients in $\mathbb{R}$.

1. Let V be a vector space over $\mathbb{R}$. Show that the following maps $T_i : V \to V$, $i = a, b, c, d$, are linear operators.

(a) $V = \mathbb{R}^{m\times m}$, $A \in \mathbb{R}^{m\times m}$ is a fixed matrix, and $T_a : \mathbb{R}^{m\times m} \to \mathbb{R}^{m\times m}$ is defined by

$$T_a(X) = AX - XA$$

for all $X \in \mathbb{R}^{m\times m}$.

Solution. Let $\alpha, \beta \in \mathbb{R}$ and $X_1, X_2 \in \mathbb{R}^{m\times m}$. Then

$$\begin{aligned} T_a(\alpha X_1 + \beta X_2) &= A(\alpha X_1 + \beta X_2) - (\alpha X_1 + \beta X_2)A \\ &= \alpha AX_1 + \beta AX_2 - \alpha X_1 A - \beta X_2 A \\ &= \alpha(AX_1 - X_1 A) + \beta(AX_2 - X_2 A) \\ &= \alpha T_a(X_1) + \beta T_b(X_2). \end{aligned}$$

(b) $V = \mathbb{R}^{m\times n}$, $A \in \mathbb{R}^{m\times m}$ and $B \in \mathbb{R}^{n\times n}$ are fixed matrices, and $T_b : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$ is defined by

$$T_b(X) = AXB$$

for all $X \in \mathbb{R}^{m\times n}$.

Solution. Let $\alpha, \beta \in \mathbb{R}$ and $X_1, X_2 \in \mathbb{R}^{m\times m}$. Then

$$\begin{aligned} T_b(\alpha X_1 + \beta X_2) &= A(\alpha X_1 + \beta X_2)B \\ &= \alpha AX_1 B + \beta AX_2 B \\ &= \alpha T_b(X_1) + \beta T_b(X_2). \end{aligned}$$

(c) $V = \mathbb{P}$, and $T_c : \mathbb{P} \to \mathbb{P}$ is defined by

$$T_c(p(x)) = p'(x)$$

for all $p(x) \in \mathbb{P}$.

Solution. Let $\alpha, \beta \in \mathbb{R}$ and $p(x), q(x) \in \mathbb{P}$. Then by the basic properties of differentiation that you have learned in high-school calculus,

$$\begin{aligned} T_c(\alpha p(x) + \beta q(x)) &= \frac{d}{dx}(\alpha p(x) + \beta q(x)) \\ &= \alpha \frac{d}{dx} p(x) + \beta \frac{d}{dx} q(x) \\ &= \alpha p'(x) + \beta q'(x) \\ &= \alpha T_c(p(x)) + \beta T_c(q(x)). \end{aligned}$$

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(d) $V = \mathbb{P}$, and $T_d : \mathbb{P} \to \mathbb{P}$ is defined by

$$T_d(p(x)) = xp(x)$$

for all $p(x) \in \mathbb{P}$.

SOLUTION. Let $\alpha, \beta \in \mathbb{R}$ and $p(x), q(x) \in \mathbb{P}$. Then by the basic properties of polynomial multiplication that you have learned in high-school algebra,

$$\begin{aligned} T_d(\alpha p(x) + \beta q(x)) &= x(\alpha p(x) + \beta q(x)) \\ &= \alpha x p(x) + \beta x q(x) \\ &= \alpha T_d(p(x)) + \beta T_d(q(x)). \end{aligned}$$

2. Let $F : \mathbb{R}^3 \to \mathbb{R}^2$, $G : \mathbb{R}^3 \to \mathbb{R}^2$, $H : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $F(x, y, z) = (y, x + z)$, $G(x, y, z) = (2z, x - y)$, $H(x, y) = (y, 2x)$ respectively.

(a) Find the formulas that define $3F - 5G$, $H \circ F$, $H \circ G$, $H \circ (F + G)$, and state the respective domains and codomains of these maps. Are these linear transformations?

SOLUTION. $3F - 5G : \mathbb{R}^3 \to \mathbb{R}^2$ and its formula is given by

$$\begin{aligned} (3F - 5G)(x, y, z) &= 3F(x, y, z) - 5G(x, y, z) \\ &= 3(y, x + z) - 5(2z, x - y) \\ &= (3y - 10z, -2x + 3z + 5y). \end{aligned}$$

$H \circ F : \mathbb{R}^3 \to \mathbb{R}^2$ and its formula is given by

$$H \circ F(x, y, z) = H(F(x, y, z)) = H(y, x + z) = (x + z, 2y).$$

$H \circ G : \mathbb{R}^3 \to \mathbb{R}^2$ and its formula is given by

$$H \circ G(x, y, z) = H(G(x, y, z)) = H(2z, x - y) = (x - y, 4z).$$

$H \circ (F + G) : \mathbb{R}^3 \to \mathbb{R}^2$ and its formula is given by

$$\begin{aligned} H \circ (F + G)(x, y, z) &= H((F + G)(x, y, z)) \\ &= H(y + 2z, 2x - y + z) \\ &= (2x - y + z, 2y + 4z). \end{aligned}$$

Alternatively, we may use Problem **3**(c) below: $H \circ (F + G) = H \circ F + H \circ G$. All four maps are linear transformations.

(b) We define $H^2 = H \circ H$, $H^3 = H \circ H \circ H$, and so on. Find the formulas that define H^2, H^3, and $H^2 \circ (F + G)$, and state the respective domains and codomains of these maps. Are these linear transformations?

SOLUTION. $H^2 : \mathbb{R}^2 \to \mathbb{R}^2$ and its formula is given by

$$H^2(x, y) = H(H(x, y)) = H(y, 2x) = (2x, 2y).$$

$H^2 : \mathbb{R}^2 \to \mathbb{R}^2$ and its formula is given by

$$H^3(x, y) = H(H^2(x, y)) = H(2x, 2y) = (2y, 4x).$$

$H^2 \circ (F + G) : \mathbb{R}^3 \to \mathbb{R}^2$ and its formula is given by

$$\begin{aligned} H^2 \circ (F + G)(x, y) &= H^2((F + G)(x, y)) \\ &= H^2(y + 2z, 2x - y + z) \\ &= (2y + 4z, 4x - 2y + 2z). \end{aligned}$$

All three maps are linear transformations.

3. Let U, W, V be finite-dimensional vector spaces over $\mathbb{F}$. Let $\alpha, \beta \in \mathbb{F}$.

(a) Let $T : V \to W$ be a linear transformation. Show that

$$\operatorname{rank}(T) \leq \dim(V).$$

SOLUTION. Let $\dim(V) = n$. Suppose $\dim(\operatorname{Im}(T)) = \operatorname{rank}(T) > n$. Then there must be at least $n+1$ vectors $\mathbf{w}_1, \ldots, \mathbf{w}_{n+1} \in \operatorname{Im}(T)$ that are linearly independent (eg. choose the first $n+1$ vectors in a basis of $\operatorname{Im}(T)$). Since $\mathbf{w}_1, \ldots, \mathbf{w}_{n+1}$ are in $\operatorname{Im}(T)$, there exist $\mathbf{v}_1, \ldots, \mathbf{v}_{n+1} \in V$ such that $T(\mathbf{v}_i) = \mathbf{w}_i$, $i = 1, \ldots, n+1$. Since $\dim(V) = n$, $\mathbf{v}_1, \ldots, \mathbf{v}_{n+1}$ must be linearly dependent, so there exist $\alpha_1, \ldots, \alpha_{n+1} \in \mathbb{F}$, not all zero, such that

$$\alpha_1 \mathbf{v}_1 + \cdots + \alpha_{n+1} \mathbf{v}_{n+1} = \mathbf{0}_V.$$

Hence we have found $\alpha_1, \ldots, \alpha_{n+1} \in \mathbb{F}$, not all zero, such that

$$T(\alpha_1 \mathbf{v}_1 + \cdots + \alpha_{n+1} \mathbf{v}_{n+1}) = T(\mathbf{0}_V),$$

ie.

$$\alpha_1 T(\mathbf{v}_1) + \cdots + \alpha_{n+1} T(\mathbf{v}_{n+1}) = \mathbf{0}_W,$$

ie.

$$\alpha_1 \mathbf{w}_1 + \cdots + \alpha_{n+1} \mathbf{w}_{n+1} = \mathbf{0}_W,$$

which implies that $\mathbf{w}_1, \ldots, \mathbf{w}_{n+1}$ are linearly dependent — a contradiction. Hence our original assumption must have been false, ie. we must have $\dim(\operatorname{Im}(T)) \leq n$.

(b) Let $S : U \to V$ and $T : V \to W$ be linear transformations. Show that

$$\operatorname{rank}(T \circ S) \leq \operatorname{rank}(T)$$

and

$$\operatorname{rank}(T \circ S) \leq \operatorname{rank}(S).$$

SOLUTION. Recall the following elementary fact from set theory (Math **74**): $A \subseteq B$ implies $f(A) \subseteq f(B)$ for any function f. In particular, since $S(U) \subseteq V$, we must have $T(S(U)) \subseteq T(V)$, ie. $T \circ S(U) \subseteq T(V)$, ie.

$$\operatorname{im}(T \circ S) \subseteq \operatorname{im}(T)$$

(recall that $\operatorname{im}(\varphi)$ is just another way of writing $\varphi(V)$ — both denote the range of φ). Now both of these are subspaces of W and so

$$\dim(\operatorname{im}(T \circ S)) \leq \dim(\operatorname{im}(T)),$$

ie.

$$\operatorname{rank}(T \circ S) \leq \operatorname{rank}(T).$$

For the second part, we define the function $\varphi : \operatorname{im}(S) \to W$ by $\varphi(\mathbf{v}) = T(\mathbf{v})$ for all $\mathbf{v} \in \operatorname{im}(S)$. Note that φ is a linear transformation since for all $\mathbf{v}_1, \mathbf{v}_2 \in \operatorname{im}(S)$ and $\alpha, \beta \in \mathbb{F}$, we have $\varphi(\alpha \mathbf{v}_1 + \beta \mathbf{v}_2) = T(\alpha \mathbf{v}_1 + \beta \mathbf{v}_2) = \alpha T(\mathbf{v}_1) + \beta T(\mathbf{v}_2) = \alpha \varphi(\mathbf{v}_1) + \beta \varphi(\mathbf{v}_2)$. So applying part (a) to φ, we get

$$\operatorname{rank}(\varphi) \leq \dim(\operatorname{im}(S)) = \operatorname{rank}(S). \tag{3.1}$$

But note that $\operatorname{im}(T \circ S) \subseteq \operatorname{im}(\varphi)$ — since if $\mathbf{w} \in \operatorname{im}(T \circ S)$, then $\mathbf{w} = T(S(\mathbf{u}))$ for some $\mathbf{u} \in U$ and therefore $\mathbf{w} = T(\mathbf{v}) = \varphi(\mathbf{v})$ where $\mathbf{v} = S(\mathbf{u}) \in \operatorname{im}(S)$; hence $\mathbf{w} \in \operatorname{im}(\varphi)$. So

$$\dim(\operatorname{im}(T \circ S)) \leq \dim(\operatorname{im}(\varphi)),$$

ie.

$$\operatorname{rank}(T \circ S) \leq \operatorname{rank}(\varphi). \tag{3.2}$$

Combining (3.1) and (3.2) then yields

$$\operatorname{rank}(T \circ S) \leq \operatorname{rank}(S).$$

(c) Let $S_1 : U \to V$, $S_2 : U \to V$, and $T : V \to W$ be linear transformations. Show that

$$T \circ (\alpha S_1 + \beta S_2) = \alpha T \circ S_1 + \beta T \circ S_2.$$

SOLUTION. Let $\mathbf{u} \in U$. Then

$$\begin{aligned} T \circ (\alpha S_1 + \beta S_2)(\mathbf{u}) &= T((\alpha S_1 + \beta S_2)(\mathbf{u})) \\ &= T(\alpha S_1(\mathbf{u}) + \beta S_2(\mathbf{u})) \\ &= \alpha T(S_1(\mathbf{u})) + \beta T(S_2(\mathbf{u})) \\ &= \alpha T \circ S_1(\mathbf{u}) + \beta T \circ S_2(\mathbf{u}). \end{aligned}$$

Since this is true for any $\mathbf{u} \in U$, we have that

$$T \circ (\alpha S_1 + \beta S_2) = \alpha T \circ S_1 + \beta T \circ S_2.$$

(d) Let $S : U \to V$, $T_1 : V \to W$, and $T_2 : V \to W$ be linear transformations. Show that

$$(\alpha T_1 + \beta T_2) \circ S = \alpha T_1 \circ S + \beta T_2 \circ S.$$

SOLUTION. Let $\mathbf{u} \in U$. Then

$$\begin{aligned} (\alpha T_1 + \beta T_2) \circ S(\mathbf{u}) &= (\alpha T_1 + \beta T_2)(S(\mathbf{u})) \\ &= \alpha T_1(S(\mathbf{u})) + \beta T_2(S(\mathbf{u})) \\ &= \alpha T_1 \circ S(\mathbf{u}) + \beta T_2 \circ S(\mathbf{u}) \end{aligned}$$

Since this is true for any $\mathbf{u} \in U$, we have that

$$(\alpha T_1 + \beta T_2) \circ S = \alpha T_1 \circ S + \beta T_2 \circ S.$$

4. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

$$T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right) = \begin{bmatrix}2\\3\\-1\end{bmatrix}, \quad T\left(\begin{bmatrix}1\\-1\\1\end{bmatrix}\right) = \begin{bmatrix}3\\0\\-2\end{bmatrix}, \quad T\left(\begin{bmatrix}-2\\7\\-1\end{bmatrix}\right) = \begin{bmatrix}2\\3\\-1\end{bmatrix}.$$

Find $\operatorname{im}(T)$, $\ker(T)$, and the formula that defines T.

SOLUTION. Let

$$\mathbf{u}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix}1\\-1\\1\end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix}-2\\7\\-1\end{bmatrix}.$$

Now if $\alpha\mathbf{u}_1 + \beta\mathbf{u}_2 + \gamma\mathbf{u}_3 = \mathbf{0}$, then solving the system gives $\alpha = \beta = \gamma = 0$. So $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly independent and since $\dim(\mathbb{R}^3) = 3$, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a basis for $\mathbb{R}^3$. Hence by a result in the lectures, we have

$$\operatorname{im}(T) = \operatorname{span}\{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3)\}.$$

We will use the Throwing-Out Algorithm to obtain a basis for $\operatorname{im}(T)$. Since $T(\mathbf{u}_1) = T(\mathbf{u}_3)$, we may throw out $T(\mathbf{u}_3)$. The remain set $\{T(\mathbf{u}_1), T(\mathbf{u}_2)\}$ is easily seen to be linearly independent:

$$\lambda \begin{bmatrix}2\\3\\-1\end{bmatrix} + \mu \begin{bmatrix}3\\0\\-2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

implies $\lambda = 0$ (just examine the second coordinate) and so $\mu = 0$. Hence $\{T(\mathbf{u}_1), T(\mathbf{u}_2)\}$ is a basis and so

$$\operatorname{im}(T) = \operatorname{span}\left\{\begin{bmatrix}2\\3\\-1\end{bmatrix}, \begin{bmatrix}3\\0\\-2\end{bmatrix}\right\}.$$

To find the formula defines T we will first need to determine how an arbitrary $\mathbf{u} = [x, y, z]^\top \in \mathbb{R}^3$ may be expressed in terms of the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. So we let

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \gamma \begin{bmatrix} -2 \\ 7 \\ -1 \end{bmatrix}$$

and solve the system to get

$$\alpha = 6x + y - 5z, \quad \beta = -7x - y + 7z, \quad \gamma = -x + z. \tag{4.3}$$

Since T is a linear operator,

$$\begin{aligned} T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) &= \alpha T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) + \beta T\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right) + \gamma T\left(\begin{bmatrix} -2 \\ 7 \\ -1 \end{bmatrix}\right) \\ &= \alpha \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + \beta \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} + \gamma \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 2(\alpha+\gamma) + 3\beta \\ 3(\alpha+\gamma) \\ -(\alpha+\gamma) - 2\beta \end{bmatrix} \\ &= \begin{bmatrix} -11x - y + 13z \\ 15x + 3y - 12z \\ 9x + y - 10z \end{bmatrix} \end{aligned}$$

upon substituting (4.3). To find $\ker(T)$, we set $T(\mathbf{u}) = \mathbf{0}$ and using the next to last expression in (4.3), we get

$$\begin{bmatrix} 2(\alpha+\gamma) + 3\beta \\ 3(\alpha+\gamma) \\ -(\alpha+\gamma) - 2\beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which gives us $\alpha = -\gamma$ and $\beta = 0$; ie. $6x + y - 5z = x - z$ and $-7x - y + 7z = 0$, ie. $x = \frac{3}{2}z$ and $y = -\frac{7}{2}z$. Hence

$$\ker(T) = \operatorname{span}\left\{\begin{bmatrix} 3 \\ -7 \\ 2 \end{bmatrix}\right\}.$$

5. Let $T : V \to W$ be a linear transformation. Let V be finite-dimensional and $\dim(V) = n$. If $\mathbf{w}_1, \ldots, \mathbf{w}_m \in W$ form a basis for $\operatorname{im}(T)$ and if $\mathbf{v}_1, \ldots, \mathbf{v}_m \in V$ are such that

$$T(\mathbf{v}_i) = \mathbf{w}_i$$

for $i = 1, \ldots, m$, show that

$$V = \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\} \oplus \ker(T).$$

Solution. Let $\mathbf{v} \in V$. Then $T(\mathbf{v}) \in \operatorname{Im}(T)$. Since $\mathbf{w}_1, \ldots, \mathbf{w}_m$ form a basis for $\operatorname{im}(T)$, there exist $\alpha_1, \ldots, \alpha_m \in \mathbb{F}$ such that

$$T(\mathbf{v}) = \alpha_1 \mathbf{w}_1 + \cdots + \alpha_m \mathbf{w}_m.$$

Since $T(\mathbf{v}_i) = \mathbf{w}_i$, we get

$$T(\mathbf{v}) = \alpha_1 T(\mathbf{v}_1) + \cdots + \alpha_m T(\mathbf{v}_m),$$

ie.

$$T(\mathbf{v}) - \alpha_1 T(\mathbf{v}_1) - \cdots - \alpha_m T(\mathbf{v}_m) = \mathbf{0}_W,$$

ie.

$$T(\mathbf{v} - \alpha_1 \mathbf{v}_1 - \cdots - \alpha_m \mathbf{v}_m) = \mathbf{0}_W$$

which implies that $\mathbf{v} - \alpha_1\mathbf{v}_1 - \cdots - \alpha_m\mathbf{v}_m \in \ker(T)$. So given any $\mathbf{v} \in V$, we could write

$$\mathbf{v} = (\alpha_1\mathbf{v}_1 + \cdots + \alpha_m\mathbf{v}_m) + (\mathbf{v} - \alpha_1\mathbf{v}_1 - \cdots - \alpha_m\mathbf{v}_m) \in \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\} + \ker(T),$$

which means that

$$V = \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\} + \ker(T).$$

We will now show that this sum is in fact a direct sum. Let $\mathbf{u}_1, \ldots, \mathbf{u}_k$ be a basis for $\ker(T)$ where $k = \operatorname{nullity}(T)$. We claim that $\{\mathbf{v}_1, \ldots, \mathbf{v}_m, \mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is linearly independent and by Homework **3**, Problem **4**(b),

$$\begin{aligned} V &= \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\} + \ker(T) \\ &= \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\} + \operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\} \\ &= \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m, \mathbf{u}_1, \ldots, \mathbf{u}_k\} \\ &= \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\} \oplus \operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\} \\ &= \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\} \oplus \ker(T). \end{aligned}$$

Suppose there exists $\lambda_1, \ldots, \lambda_m, \mu_1, \ldots, \mu_k \in \mathbb{F}$ such that

$$\lambda_1\mathbf{v}_1 + \cdots + \lambda_m\mathbf{v}_m + \mu_1\mathbf{u}_1 + \cdots + \mu_k\mathbf{u}_k = \mathbf{0}_V. \tag{5.4}$$

Then applying T to both sides of the equation gives

$$\lambda_1 T(\mathbf{v}_1) + \cdots + \lambda_m T(\mathbf{v}_m) + \mu_1 T(\mathbf{u}_1) + \cdots + \mu_k T(\mathbf{u}_k) = \mathbf{0}_W.$$

Now since $T(\mathbf{u}_i) = \mathbf{0}_W$ for all $i = 1, \ldots, k$, and $T(\mathbf{v}_j) = \mathbf{w}_j$ for all $j = 1, \ldots, m$, we get

$$\lambda_1\mathbf{w}_1 + \cdots + \lambda_m\mathbf{w}_m = \mathbf{0}_W.$$

As $\mathbf{w}_1, \ldots, \mathbf{w}_m$ are linearly independent, we get

$$\lambda_1 = \cdots = \lambda_m = 0.$$

So we are left with

$$\mu_1\mathbf{u}_1 + \cdots + \mu_k\mathbf{u}_k = \mathbf{0}_V$$

in (5.4). But since $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are also linearly independent, we then get

$$\mu_1 = \cdots = \mu_k = 0.$$

So $\mathbf{v}_1, \ldots, \mathbf{v}_m, \mathbf{u}_1, \ldots, \mathbf{u}_k$ are linearly independent.