You may use without proof any results from the lectures as well as any basic properties of matrix operations and calculus operations. \( \mathbb{P} \) will denote the vector space of polynomials in \( x \) with coefficients in \( \mathbb{R} \).

1. Let \( V \) be a vector space over \( \mathbb{R} \). Show that the following maps \( T_i : V \to V, \ i = a,b,c,d \), are linear operators.

(a) \( V = \mathbb{R}^{m \times m} \), \( A \in \mathbb{R}^{m \times m} \) is a fixed matrix, and \( T_a : \mathbb{R}^{m \times m} \to \mathbb{R}^{m \times m} \) is defined by

\[
T_a(X) = AX - XA
\]

for all \( X \in \mathbb{R}^{m \times m} \).

**Solution.** Let \( \alpha, \beta \in \mathbb{R} \) and \( X_1, X_2 \in \mathbb{R}^{m \times m} \). Then

\[
T_a(\alpha X_1 + \beta X_2) = A(\alpha X_1 + \beta X_2) - (\alpha X_1 + \beta X_2)A
\]

\[
= \alpha AX_1 + \beta AX_2 - \alpha X_1 A - \beta X_2 A
\]

\[
= \alpha (AX_1 - X_1 A) + \beta (AX_2 - X_2 A)
\]

\[
= \alpha T_a(X_1) + \beta T_a(X_2).
\]

(b) \( V = \mathbb{R}^{m \times n} \), \( A \in \mathbb{R}^{m \times m} \) and \( B \in \mathbb{R}^{n \times n} \) are fixed matrices, and \( T_b : \mathbb{R}^{m \times n} \to \mathbb{R}^{m \times n} \) is defined by

\[
T_b(X) = AXB
\]

for all \( X \in \mathbb{R}^{m \times n} \).

**Solution.** Let \( \alpha, \beta \in \mathbb{R} \) and \( X_1, X_2 \in \mathbb{R}^{m \times m} \). Then

\[
T_b(\alpha X_1 + \beta X_2) = A(\alpha X_1 + \beta X_2)B
\]

\[
= \alpha AX_1 B + \beta AX_2 B
\]

\[
= \alpha T_b(X_1) + \beta T_b(X_2).
\]

(c) \( V = \mathbb{P} \), and \( T_c : \mathbb{P} \to \mathbb{P} \) is defined by

\[
T_c(p(x)) = p'(x)
\]

for all \( p(x) \in \mathbb{P} \).

**Solution.** Let \( \alpha, \beta \in \mathbb{R} \) and \( p(x), q(x) \in \mathbb{P} \). Then by the basic properties of differentiation that you have learned in high-school calculus,

\[
T_c(\alpha p(x) + \beta q(x)) = \frac{d}{dx} (\alpha p(x) + \beta q(x))
\]

\[
= \alpha \frac{d}{dx} p(x) + \beta \frac{d}{dx} q(x)
\]

\[
= \alpha p'(x) + \beta q'(x)
\]

\[
= \alpha T_c(p(x)) + \beta T_c(q(x)).
\]
(d) \( V = \mathbb{P} \), and \( T_d : \mathbb{P} \to \mathbb{P} \) is defined by
\[
T_d(p(x)) = xp(x)
\]
for all \( p(x) \in \mathbb{P} \).

**Solution.** Let \( \alpha, \beta \in \mathbb{R} \) and \( p(x), q(x) \in \mathbb{P} \). Then by the basic properties of polynomial multiplication that you have learned in high-school algebra,
\[
T_d(\alpha p(x) + \beta q(x)) = x(\alpha p(x) + \beta q(x))
= \alpha xp(x) + \beta xq(x)
= \alpha T_d(p(x)) + \beta T_d(q(x)).
\]

2. Let \( F : \mathbb{R}^3 \to \mathbb{R}^2 \), \( G : \mathbb{R}^3 \to \mathbb{R}^2 \), \( H : \mathbb{R}^2 \to \mathbb{R}^2 \) be defined by \( F(x, y, z) = (y, x + z) \), \( G(x, y, z) = (2z, x - y) \), \( H(x, y) = (y, 2x) \) respectively.

(a) Find the formulas that define \( 3F - 5G, H \circ F, H \circ G, H \circ (F + G) \), and state the respective domains and codomains of these maps. Are these linear transformations?

**Solution.** \( 3F - 5G : \mathbb{R}^3 \to \mathbb{R}^2 \) and its formula is given by
\[
(3F - 5G)(x, y, z) = 3F(x, y, z) - 5G(x, y, z)
= 3(y, x + z) - 5(2z, x - y)
= (3y - 10z, 2x + 3z + 5y).
\]

\( H \circ F : \mathbb{R}^3 \to \mathbb{R}^2 \) and its formula is given by
\[
H \circ F(x, y, z) = H(F(x, y, z)) = H(y, x + z) = (x + z, 2y).
\]

\( H \circ G : \mathbb{R}^3 \to \mathbb{R}^2 \) and its formula is given by
\[
H \circ G(x, y, z) = H(G(x, y, z)) = H(2z, x - y) = (x - y, 4z).
\]

\( H \circ (F + G) : \mathbb{R}^3 \to \mathbb{R}^2 \) and its formula is given by
\[
H \circ (F + G)(x, y, z) = H((F + G)(x, y, z))
= H(y + 2z, 2x - y + z)
= (2x - y + z, 2y + 4z).
\]

Alternatively, we may use Problem 3(c) below: \( H \circ (F + G) = H \circ F + H \circ G \). All four maps are linear transformations.

(b) We define \( H^2 = H \circ H \), \( H^3 = H \circ H \circ H \), and so on. Find the formulas that define \( H^2 \), \( H^3 \), and \( H^2 \circ (F + G) \), and state the respective domains and codomains of these maps. Are these linear transformations?

**Solution.** \( H^2 : \mathbb{R}^2 \to \mathbb{R}^2 \) and its formula is given by
\[
H^2(x, y) = H(H(x, y)) = H(y, 2x) = (2x, 2y).
\]

\( H^3(x, y) = H(H^3(x, y)) = H(2x, 2y) = (2y, 4x) \).

\( H^2 \circ (F + G) : \mathbb{R}^3 \to \mathbb{R}^2 \) and its formula is given by
\[
H^2 \circ (F + G)(x, y) = H^2((F + G)(x, y))
= H^2(y + 2z, 2x - y + z)
= (2y + 4z, 4x - 2y + 2z).
\]

All three maps are linear transformations.

3. Let \( U, W, V \) be finite-dimensional vector spaces over \( \mathbb{F} \). Let \( \alpha, \beta \in \mathbb{F} \).
(a) Let $T : V \to W$ be a linear transformation. Show that
\[
\text{rank}(T) \leq \dim(V).
\]
**Solution.** Let $\dim(V) = n$. Suppose $\dim(\text{Im}(T)) = \text{rank}(T) > n$. Then there must be at least $n + 1$ vectors $w_1, \ldots, w_{n+1} \in \text{Im}(T)$ that are linearly independent (e.g. choose the first $n + 1$ vectors in a basis of $\text{Im}(T)$). Since $w_1, \ldots, w_{n+1}$ are in $\text{Im}(T)$, there exist $v_1, \ldots, v_{n+1} \in V$ such that $T(v_i) = w_i$, $i = 1, \ldots, n+1$. Since $\dim(V) = n$, $v_1, \ldots, v_{n+1}$ must be linearly dependent, so there exist $\alpha_1, \ldots, \alpha_{n+1} \in F$, not all zero, such that
\[
\alpha_1 v_1 + \cdots + \alpha_{n+1} v_{n+1} = 0_V.
\]
Hence we have found $\alpha_1, \ldots, \alpha_{n+1} \in F$, not all zero, such that
\[
T(\alpha_1 v_1 + \cdots + \alpha_{n+1} v_{n+1}) = T(0_V),
\]
\[
\alpha_1 T(v_1) + \cdots + \alpha_{n+1} T(v_{n+1}) = 0_W,
\]
\[
\alpha_1 w_1 + \cdots + \alpha_{n+1} w_{n+1} = 0_W,
\]
which implies that $w_1, \ldots, w_{n+1}$ are linearly dependent — a contradiction. Hence our original assumption must have been false, i.e. we must have $\dim(\text{Im}(T)) \leq n$.

(b) Let $S : U \to V$ and $T : V \to W$ be linear transformations. Show that
\[
\text{rank}(T \circ S) \leq \text{rank}(T)
\]
and
\[
\text{rank}(T \circ S) \leq \text{rank}(S).
\]
**Solution.** Recall the following elementary fact from set theory (Math 74): $A \subseteq B$ implies $f(A) \subseteq f(B)$ for any function $f$. In particular, since $S(U) \subseteq V$, we must have $T(S(U)) \subseteq T(V)$, i.e. $T \circ S(U) \subseteq T(V)$, i.e.
\[
\text{im}(T \circ S) \subseteq \text{im}(T)
\]
(recall that $\text{im}(\varphi)$ is just another way of writing $\varphi(V)$ — both denote the range of $\varphi$). Now both of these are subspaces of $W$ and so
\[
\dim(\text{im}(T \circ S)) \leq \dim(\text{im}(T)),
\]
\[
\text{rank}(T \circ S) \leq \text{rank}(T).
\]
For the second part, we define the function $\varphi : \text{im}(S) \to W$ by $\varphi(v) = T(v)$ for all $v \in \text{im}(S)$. Note that $\varphi$ is a linear transformation since for all $v_1, v_2 \in \text{im}(S)$ and $\alpha, \beta \in F$, we have $\varphi(\alpha v_1 + \beta v_2) = T(\alpha v_1 + \beta v_2) = \alpha T(v_1) + \beta T(v_2) = \alpha \varphi(v_1) + \beta \varphi(v_2)$. So applying part (a) to $\varphi$, we get
\[
\text{rank}(\varphi) \leq \dim(\text{im}(S)) = \text{rank}(S).
\]
But note that $\text{im}(T \circ S) \subseteq \text{im}(\varphi)$ — since if $w \in \text{im}(T \circ S)$, then $w = T(S(u))$ for some $u \in U$ and therefore $w = T(v) = \varphi(v)$ where $v = S(u) \in \text{im}(S)$; hence $w \in \text{im}(\varphi)$. So
\[
\text{dim}(\text{im}(T \circ S)) \leq \text{dim}(\text{im}(\varphi)),
\]
\[
\text{rank}(T \circ S) \leq \text{rank}(\varphi).
\]
Combining (3.1) and (3.2) then yields
\[
\text{rank}(T \circ S) \leq \text{rank}(S).
\]
(c) Let $S_1 : U \rightarrow V$, $S_2 : U \rightarrow V$, and $T : V \rightarrow W$ be linear transformations. Show that
\[ T \circ (\alpha S_1 + \beta S_2) = \alpha T \circ S_1 + \beta T \circ S_2. \]

**Solution.** Let $u \in U$. Then
\begin{align*}
T \circ (\alpha S_1 + \beta S_2)(u) &= T((\alpha S_1 + \beta S_2)(u)) \\
&= T(\alpha S_1(u) + \beta S_2(u)) \\
&= \alpha T(S_1(u)) + \beta T(S_2(u)) \\
&= \alpha T \circ S_1(u) + \beta T \circ S_2(u).
\end{align*}

Since this is true for any $u \in U$, we have that
\[ T \circ (\alpha S_1 + \beta S_2) = \alpha T \circ S_1 + \beta T \circ S_2. \]

(d) Let $S : U \rightarrow V$, $T_1 : V \rightarrow W$, and $T_2 : V \rightarrow W$ be linear transformations. Show that
\[ (\alpha T_1 + \beta T_2) \circ S = \alpha T_1 \circ S + \beta T_2 \circ S. \]

**Solution.** Let $u \in U$. Then
\begin{align*}
(\alpha T_1 + \beta T_2) \circ S(u) &= (\alpha T_1 + \beta T_2)(S(u)) \\
&= \alpha T_1(S(u)) + \beta T_2(S(u)) \\
&= \alpha T_1 \circ S(u) + \beta T_2 \circ S(u)
\end{align*}

Since this is true for any $u \in U$, we have that
\[ (\alpha T_1 + \beta T_2) \circ S = \alpha T_1 \circ S + \beta T_2 \circ S. \]

4. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear transformation such that
\[ T \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, \quad T \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}, \quad T \begin{pmatrix} -2 \\ 7 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}. \]

Find $\text{im}(T)$, $\ker(T)$, and the formula that defines $T$.

**Solution.** Let
\[ u_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} -2 \\ 7 \\ -1 \end{pmatrix}. \]

Now if $\alpha u_1 + \beta u_2 + \gamma u_3 = \mathbf{0}$, then solving the system gives $\alpha = \beta = \gamma = 0$. So $u_1, u_2, u_3$ are linearly independent and since $\dim(\mathbb{R}^3) = 3$, $\{u_1, u_2, u_3\}$ is a basis for $\mathbb{R}^3$. Hence by a result in the lectures, we have
\[ \text{im}(T) = \text{span}\{T(u_1), T(u_2), T(u_3)\}. \]

We will use the Throwing-Out Algorithm to obtain a basis for $\text{im}(T)$. Since $T(u_1) = T(u_2)$, we may throw out $T(u_3)$. The remain set $\{T(u_1), T(u_2)\}$ is easily seen to be linearly independent:
\[ \lambda \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \]

implies $\lambda = 0$ (just examine the second coordinate) and so $\mu = 0$. Hence $\{T(u_1), T(u_2)\}$ is a basis and so
\[ \text{im}(T) = \text{span}\left\{ \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} \right\}. \]
To find the formula defines $T$ we will first need to determine how an arbitrary $u = [x, y, z]^\top \in \mathbb{R}^3$ may be expressed in terms of the basis $\{u_1, u_2, u_3\}$. So we let
\[
\begin{bmatrix}
x \\
y \\
z \\
\end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \gamma \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}
\]
and solve the system to get
\[
\alpha = 6x + y - 5z, \quad \beta = -7x - y + 7z, \quad \gamma = -x + z.
\]
(4.3)
Since $T$ is a linear operator,
\[
T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \alpha T \left( \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right) + \beta T \left( \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right) + \gamma T \left( \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} \right)
\]
\[
= \alpha \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + \beta \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} + \gamma \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}
\]
\[
= \begin{bmatrix} 2(\alpha + \gamma) + 3\beta \\ 3(\alpha + \gamma) \\ -(\alpha + \gamma) - 2\beta \end{bmatrix}
\]
\[
= \begin{bmatrix} 2(\alpha + \gamma) + 3\beta \\ 3(\alpha + \gamma) \\ -(\alpha + \gamma) - 2\beta \end{bmatrix}
\]
\[
= \begin{bmatrix} 2(\alpha + \gamma) + 3\beta \\ 3(\alpha + \gamma) \\ -(\alpha + \gamma) - 2\beta \end{bmatrix}
\]
upon substituting (4.3). To find $\ker(T)$, we set $T(u) = 0$ and using the next to last expression in (4.3), we get
\[
\begin{bmatrix} 2(\alpha + \gamma) + 3\beta \\ 3(\alpha + \gamma) \\ -(\alpha + \gamma) - 2\beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]
which gives us $\alpha = -\gamma$ and $\beta = 0$; i.e. $6x + y - 5z = x - z$ and $-7x - y + 7z = 0$, i.e. $x = \frac{3}{2} z$ and $y = -\frac{7}{2} z$. Hence
\[
\ker(T) = \text{span}\left\{\begin{bmatrix} 3 \\ 7 \\ -2 \end{bmatrix}\right\}.
\]
5. Let $T : V \to W$ be a linear transformation. Let $V$ be finite-dimensional and $\dim(V) = n$. If $w_1, \ldots, w_m \in W$ form a basis for $\text{im}(T)$ and if $v_1, \ldots, v_m \in V$ are such that $T(v_i) = w_i$ for $i = 1, \ldots, m$, show that $V = \text{span}\{v_1, \ldots, v_m\} \oplus \ker(T)$.

**Solution.** Let $v \in V$. Then $T(v) \in \text{Im}(T)$. Since $w_1, \ldots, w_m$ form a basis for $\text{im}(T)$, there exist $\alpha_1, \ldots, \alpha_m \in \mathbb{F}$ such that
\[
T(v) = \alpha_1 w_1 + \cdots + \alpha_m w_m.
\]
Since $T(v_i) = w_i$, we get
\[
T(v) = \alpha_1 T(v_1) + \cdots + \alpha_m T(v_m),
\]
ie.
\[
T(v) - \alpha_1 T(v_1) - \cdots - \alpha_m T(v_m) = 0_W,
\]
ie.
\[
T(v - \alpha_1 v_1 - \cdots - \alpha_m v_m) = 0_W
\]
which implies that \( \mathbf{v} - \alpha_1 \mathbf{v}_1 - \cdots - \alpha_m \mathbf{v}_m \in \ker(T) \). So given any \( \mathbf{v} \in V \), we could write

\[
\mathbf{v} = (\alpha_1 \mathbf{v}_1 + \cdots + \alpha_m \mathbf{v}_m) + (\mathbf{v} - \alpha_1 \mathbf{v}_1 - \cdots - \alpha_m \mathbf{v}_m) \in \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\} + \ker(T),
\]

which means that

\[
V = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\} + \ker(T).
\]

We will now show that this sum is in fact a direct sum. Let \( \mathbf{u}_1, \ldots, \mathbf{u}_k \) be a basis for \( \ker(T) \) where \( k = \text{nullity}(T) \). We claim that \( \{\mathbf{v}_1, \ldots, \mathbf{v}_m, \mathbf{u}_1, \ldots, \mathbf{u}_k\} \) is linearly independent and by Homework 3, Problem 4(b),

\[
V = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\} + \ker(T)
\]

\[
= \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\} + \text{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}
\]

\[
= \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m, \mathbf{u}_1, \ldots, \mathbf{u}_k\}
\]

\[
= \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\} \oplus \text{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}
\]

\[
= \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\} \oplus \ker(T).
\]

Suppose there exists \( \lambda_1, \ldots, \lambda_m, \mu_1, \ldots, \mu_k \in \mathbb{F} \) such that

\[
\lambda_1 \mathbf{v}_1 + \cdots + \lambda_m \mathbf{v}_m + \mu_1 \mathbf{u}_1 + \cdots + \mu_k \mathbf{u}_k = \mathbf{0}_V. \tag{5.4}
\]

Then applying \( T \) to both sides of the equation gives

\[
\lambda_1 T(\mathbf{v}_1) + \cdots + \lambda_m T(\mathbf{v}_m) + \mu_1 T(\mathbf{u}_1) + \cdots + \mu_k T(\mathbf{u}_k) = \mathbf{0}_W.
\]

Now since \( T(\mathbf{u}_i) = \mathbf{0}_W \) for all \( i = 1, \ldots, k \), and \( T(\mathbf{v}_j) = \mathbf{w}_j \) for all \( j = 1, \ldots, m \), we get

\[
\lambda_1 \mathbf{w}_1 + \cdots + \lambda_m \mathbf{w}_m = \mathbf{0}_W.
\]

As \( \mathbf{w}_1, \ldots, \mathbf{w}_m \) are linearly independent, we get

\[
\lambda_1 = \cdots = \lambda_m = 0.
\]

So we are left with

\[
\mu_1 \mathbf{u}_1 + \cdots + \mu_k \mathbf{u}_k = \mathbf{0}_V
\]

in (5.4). But since \( \mathbf{u}_1, \ldots, \mathbf{u}_k \) are also linearly independent, we then get

\[
\mu_1 = \cdots = \mu_k = 0.
\]

So \( \mathbf{v}_1, \ldots, \mathbf{v}_m, \mathbf{u}_1, \ldots, \mathbf{u}_k \) are linearly independent.