MATH 110: LINEAR ALGEBRA
FALL 2007/08
PROBLEM SET 5

1. Let W_1 and W_2 be the following subspaces of $\mathbb{R}^4$.

$$W_1 = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}, \quad W_2 = \operatorname{span}\left\{ \begin{bmatrix} 2 \\ 5 \\ -6 \\ -5 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -7 \\ 3 \end{bmatrix} \right\}.$$

(a) Find a basis and state the dimension of $W_1 \cap W_2$.

SOLUTION. Let the three vectors spanning W_1 be $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ and the two vectors spanning W_2 be $\mathbf{v}_1, \mathbf{v}_2$ respectively. If $\mathbf{w} \in W_1 \cap W_2$, then there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ and $\beta_1, \beta_2 \in \mathbb{R}$ such that

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 = \mathbf{w} = \beta_1 \mathbf{v}_1 + \beta_2 \mathbf{v}_2.$$

In other words, we want to solve

$$\alpha_1 \begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \beta_1 \begin{bmatrix} 2 \\ 5 \\ -6 \\ -5 \end{bmatrix} + \beta_2 \begin{bmatrix} -1 \\ 2 \\ -7 \\ 3 \end{bmatrix}.$$

Forming the augmented system to solve for $\alpha_1, \alpha_2, \alpha_3$ in terms of β_1, β_2 (alternatively, we could also solve for β_1 and β_2 in terms of $\alpha_1, \alpha_2, \alpha_3$), we get upon Gauss-Jordan elimination,

$$\left[\begin{array}{rrr|r} 1 & 3 & -1 & 2\beta_1 - \beta_2 \\ 2 & 1 & 0 & 5\beta_1 + 2\beta_2 \\ -1 & 1 & 1 & -6\beta_1 - 7\beta_2 \\ -2 & 1 & -1 & -5\beta_1 + 3\beta_2 \end{array}\right] \longrightarrow \dots \longrightarrow \left[\begin{array}{rrr|r} 1 & 0 & 0 & 3\beta_1 + 2\beta_2 \\ 0 & 1 & 0 & -\beta_1 - 2\beta_2 \\ 0 & 0 & 1 & -2\beta_1 - 9\beta_2 \\ 0 & 0 & 0 & -6\beta_2 \end{array}\right].$$

For this system to be consistent, we must have $\beta_2 = 0$. Hence

$$\mathbf{w} = \beta_1 [2, 5, -6, -5]^\top$$

for some $\beta_2 \in \mathbb{R}$. In other words,

$$W_1 \cap W_2 = \operatorname{span}\{[2, 5, -6, -5]^\top\}$$

and so $\dim(W_1 \cap W_2) = 1$. As a sanity check, observe that $\beta_2 = 0$ gives

$$\alpha_1 = 3\beta_1, \quad \alpha_2 = -\beta_1, \quad \alpha_3 = -2\beta_1$$

and indeed $\mathbf{w}$ can also be expressed as

$$\mathbf{w} = 3\beta_1 \begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix} - \beta_1 \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix} - 2\beta_1 \begin{bmatrix} -1 \\ 0 \\ 1 \\ -1 \end{bmatrix} \in W_1.$$

(b) Find a basis and state the dimension of $W_1 + W_2$.

SOLUTION. Observe that

$$\operatorname{span}(S_1) + \operatorname{span}(S_2) = \operatorname{span}(S_1 \cup S_2).$$

So

$$W_1 + W_2 = \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{v}_1, \mathbf{v}_2\}.$$

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We will apply the Throwing-Out Algorithm to get a basis. First we try to solve

$$\mathbf{v}_2 = \alpha\mathbf{u}_1 + \beta\mathbf{u}_2 + \gamma\mathbf{u}_3 + \delta\mathbf{v}_1. \tag{1.1}$$

Applying Gauss-Jordan Elimination, we get

$$\left[\begin{array}{rrrr|r} 1 & 3 & -1 & 2 & -1 \\ 2 & 1 & 0 & 5 & 2 \\ -1 & 1 & 1 & -6 & -7 \\ -2 & 1 & -1 & -5 & 3 \end{array}\right] \longrightarrow \cdots \longrightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 0 & -3 & 2 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & 2 & -9 \\ 0 & 0 & 0 & 0 & -6 \end{array}\right]$$

which shows that (1.1) has no solution — so we cannot discard $\mathbf{v}_2$. Next we try to solve

$$\mathbf{v}_1 = \alpha\mathbf{u}_1 + \beta\mathbf{u}_2 + \gamma\mathbf{u}_3 + \delta\mathbf{v}_2. \tag{1.2}$$

Applying Gauss-Jordan Elimination, we get

$$\left[\begin{array}{rrr|r} 1 & 3 & -1 & 2 \\ 2 & 1 & 2 & 5 \\ -1 & 1 & -7 & -6 \\ -2 & 1 & 3 & -5 \end{array}\right] \longrightarrow \cdots \longrightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

which shows that (1.2) has a solution (given by $\alpha = 3, \beta = -1, \gamma = -2, \delta = 0$). Hence we may discard $\mathbf{v}_1$. Next we try to solve

$$\mathbf{u}_3 = \alpha\mathbf{u}_1 + \beta\mathbf{u}_2 + \gamma\mathbf{v}_2. \tag{1.3}$$

Applying Gauss-Jordan Elimination, we get

$$\left[\begin{array}{rrr|r} 1 & 3 & -1 & -1 \\ 2 & 1 & 2 & 0 \\ -1 & 1 & -7 & 1 \\ -2 & 1 & 3 & -1 \end{array}\right] \longrightarrow \cdots \longrightarrow \left[\begin{array}{rrr|r} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

which shows that (1.3) has no solution — so we cannot discard $\mathbf{u}_3$. Next we try to solve

$$\mathbf{u}_2 = \alpha\mathbf{u}_1 + \beta\mathbf{u}_3 + \gamma\mathbf{v}_2. \tag{1.4}$$

Applying Gauss-Jordan Elimination, we get

$$\left[\begin{array}{rrr|r} 1 & -1 & -1 & 3 \\ 2 & 0 & 2 & 1 \\ -1 & 1 & -7 & 1 \\ -2 & -1 & 3 & 1 \end{array}\right] \longrightarrow \cdots \longrightarrow \left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{array}\right]$$

which shows that (1.4) has no solution — so we cannot discard $\mathbf{u}_2$. Lastly we try to solve

$$\mathbf{u}_1 = \alpha\mathbf{u}_2 + \beta\mathbf{u}_3 + \gamma\mathbf{v}_2. \tag{1.5}$$

Applying Gauss-Jordan Elimination, we get

$$\left[\begin{array}{rrr|r} 3 & -1 & -1 & 1 \\ 1 & 0 & 2 & 2 \\ 1 & 1 & -7 & -1 \\ 1 & -1 & 3 & -2 \end{array}\right] \longrightarrow \cdots \longrightarrow \left[\begin{array}{rrr|r} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -\frac{11}{2} \\ 0 & 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 & 11 \end{array}\right]$$

which shows that (1.5) has no solution — so we cannot discard $\mathbf{u}_1$. Hence by the Throwing-Out Algorithm the set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{v}_2\}$ is a basis for $W_1 + W_2$ and so $\dim(W_1 + W_2) = 4$. [Note: We will soon learn a more efficient method to determine a basis from a spanning set of vectors.]

2. Let $\mathbb{F}$ be a field. Let W_1 and W_2 be the following subspaces of $\mathbb{F}^n$.

$$W_1 = \{(x_1, x_2, \ldots, x_n) \in \mathbb{F}^n \mid x_1 + x_2 + \cdots + x_n = 0\},$$
$$W_2 = \{(x_1, x_2, \ldots, x_n) \in \mathbb{F}^n \mid x_1 = x_2 = \cdots = x_n\}.$$

(a) Find a basis and state the dimension of W_1.
SOLUTION. Let $(x_1, x_2, \ldots, x_n) \in W_1$. Then

$$(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, -x_1 - x_2 - \cdots - x_{n-1}).$$

Note that

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ -x_1 - x_2 - \cdots - x_{n-1} \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ -1 \end{bmatrix} + \cdots + x_{n-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ -1 \end{bmatrix}$$

and so the vectors $\mathbf{u}_1 = (1, 0, \ldots, 0, -1), \mathbf{u}_2 = (0, 1, \ldots, 0, -1), \ldots, \mathbf{u}_{n-1} = (0, 0, \ldots, 1, -1)$ span W_1. If

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_{n-1} \mathbf{u}_{n-1} = \mathbf{0},$$

then $(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, -\alpha_1 - \alpha_2 - \cdots - \alpha_{n-1}) = (0, 0, \ldots, 0, 0)$ and so $\alpha_1 = \alpha_2 = \cdots = \alpha_{n-1} = 0$. Hence $\mathbf{u}_1, \ldots, \mathbf{u}_{n-1}$ are linearly independent. So $\{\mathbf{u}_1, \ldots, \mathbf{u}_{n-1}\}$ is a basis for W_1 and $\dim(W_1) = n - 1$.

(b) Find a basis and state the dimension of W_2.
SOLUTION. Let $(x_1, x_2, \ldots, x_n) \in W_2$. Then

$$(x_1, x_2, \ldots, x_n) = (x_1, x_1, \ldots, x_1) = x_1(1, 1, \ldots, 1).$$

So the vector $\mathbf{u} = (1, 1, \ldots, 1)$ spans W_2. If $\alpha \mathbf{u} = \mathbf{0}$,then $\alpha = 0$ since $\mathbf{u} \neq \mathbf{0}$. Hence $\{\mathbf{u}\}$ is a basis for W_2 and $\dim(W_2) = 1$.

(c) Show that $\mathbb{R}^n = W_1 \oplus W_2$.
SOLUTION. Let $\mathbf{w} \in W_1 \cap W_2$. Then $\mathbf{w} \in W_2$ implies $\mathbf{w} = (\alpha, \alpha, \ldots, \alpha)$ but then $\mathbf{w} \in W_1$ implies that $\alpha + \alpha + \cdots + \alpha = 0$, so $n\alpha = 0$, so $\alpha = 0$, and so $\mathbf{w} = \mathbf{0}$. By Homework **4**, Problem **4**(a),

$$\dim(W_1 + W_2) + \dim(W_1 \cap W_2) = \dim(W_1) + \dim(W_2) = n - 1 + 1 = n.$$

Since $\dim(W_1 \cap W_2) = \dim\{\mathbf{0}\} = 0$, we have that $\dim(W_1 + W_2) = n = \dim(\mathbb{R}^n)$. Since $W_1 + W_2$ is a subspace of $\mathbb{R}^n$, we get $W_1 + W_2 = \mathbb{R}^n$. [Note: This statement is not true in general. For example, if $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$, ie. the field of two elements $\{0, 1\}$ under binary arithmetic, then $\mathbb{F}^2 \neq W_1 \oplus W_2$ since $W_1 \cap W_2 \neq \{\mathbf{0}_{\mathbb{F}^2}\}$. Why? Because $(1, 1) \in W_1$ (recall that $1 + 1 = 0$ in $\mathbb{F}$) and $(1, 1) \in W_2$ but $\mathbf{0}_{\mathbb{F}^2} = (0, 0) \neq (1, 1)$.]

3. Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. Which of the following functions $T_i : \mathbb{R}^2 \to \mathbb{R}^2$ $(i = a, b, c, d)$ are linear? Justify your answers.
(a) $T_a(x, y) = (\alpha x + \beta y, \gamma x + \delta y)$,
(b) $T_b(x, y) = (\alpha x^2 + \beta y^2, \gamma x^2 + \delta y^2)$,
(c) $T_c(x, y) = (\alpha^2 x + \beta^2 y, \gamma^2 x + \delta^2 y)$,
(d) $T_d(x, y) = (\alpha x + \beta, \gamma + \delta y)$,
where $(x, y) \in \mathbb{R}^2$.
SOLUTION. T_a and T_c are linear. Let $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ and $\lambda, \mu \in \mathbb{R}$. Then

$$\begin{aligned}
T_a(\lambda(x_1, y_1) + \mu(x_2, y_2)) &= T_a(\lambda x_1 + \mu x_2, \lambda y_1 + \mu y_2) \\
&= (\alpha(\lambda x_1 + \mu x_2) + \beta(\lambda y_1 + \mu y_2), \gamma(\lambda x_1 + \mu x_2) + \delta(\lambda y_1 + \mu y_2)) \\
&= (\alpha(\lambda x_1 + \mu x_2) + \beta(\lambda y_1 + \mu y_2), \gamma(\lambda x_1 + \mu x_2) + \delta(\lambda y_1 + \mu y_2)) \\
&= (\alpha\lambda x_1 + \alpha\mu x_2 + \beta\lambda y_1 + \beta\mu y_2, \gamma\lambda x_1 + \gamma\mu x_2 + \delta\lambda y_1 + \delta\mu y_2) \\
&= (\lambda\alpha x_1 + \lambda\beta y_1 + \mu\alpha x_2 + \mu\beta y_2, \lambda\gamma x_1 + \lambda\delta y_1 + \mu\gamma x_2 + \mu\delta y_2) \\
&= (\lambda(\alpha x_1 + \beta y_1) + \mu(\alpha x_2 + \beta y_2), \lambda(\gamma x_1 + \delta y_1) + \mu(\gamma x_2 + \delta y_2)) \\
&= \lambda(\alpha x_1 + \beta y_1, \gamma x_1 + \delta y_1) + \mu(\alpha x_2 + \beta y_2, \gamma x_2 + \delta y_2) \\
&= \lambda T_a(x_1, y_1) + \mu T_a(x_2, y_2).
\end{aligned}$$

T_c is just T_a with $\alpha, \beta, \gamma, \delta$ (which can be any constants in $\mathbb{R}$) being $\alpha^2, \beta^2, \gamma^2, \delta^2$. T_b is not linear unless $\alpha = \beta = \gamma = \delta = 0$. Say, suppose $\gamma \neq 0$, then

$$T_b(-(1,0)) = T_b(-1,0) = (\alpha, \gamma) \neq (-\alpha, -\gamma) = -T_b(1,0)$$

since $\gamma \neq -\gamma$. T_d is not linear unless $\beta = \gamma = 0$ since

$$T_d(0,0) = (\beta, \gamma) \neq (0,0)$$

if $\beta \neq 0$ or $\gamma \neq 0$.

4. Let $\mathbb{P}$ be the vector space of polynomials with coefficients in $\mathbb{R}$. Which of the following functions $S_i : \mathbb{P} \to \mathbb{P}$ $(i = a, b, c, d)$ are linear? Justify your answers.
(a) $S_a(p(x)) = p(x^2)$,
(b) $S_b(p(x)) = [p(x)]^2$,
(c) $S_c(p(x)) = x^2 p(x)$,
(d) $S_d(p(x)) = p(x+2)$,
where $p(x) \in \mathbb{P}$.

SOLUTION. S_a, S_c, and S_d are linear. Let $p(x), q(x) \in \mathbb{P}$ and $\lambda, \mu \in \mathbb{R}$. Let

$$d = \max\{\deg(p(x)), \deg(q(x))\}.$$

Then we may write[1]

$$p(x) = a_0 + a_1 x + \cdots + a_d x^d \qquad \text{and} \qquad q(x) = b_0 + b_1 x + \cdots + b_d x^d.$$

S_a is linear because

$$\begin{aligned}
S_a(\lambda p(x) + \mu q(x)) &= S_a(\lambda(a_0 + a_1 x + a_2 x^2 + \cdots + a_d x^d) + \mu(b_0 + b_1 x + b_2 x^2 + \cdots + b_d x^d)) \\
&= S_a((\lambda a_0 + \mu b_0) + (\lambda a_1 + \mu b_1)x + (\lambda a_2 + \mu b_2)x^2 + \cdots + (\lambda a_d + \mu b_d)x^d) \\
&= (\lambda a_0 + \mu b_0) + (\lambda a_1 + \mu b_1)x^2 + (\lambda a_2 + \mu b_2)x^4 + \cdots + (\lambda a_d + \mu b_d)x^{2d} \\
&= \lambda(a_0 + a_1 x^2 + a_2 x^4 + \cdots + a_d x^{2d}) + \mu(b_0 + b_1 x^2 + b_2 x^4 + \cdots + b_d x^{2d}) \\
&= \lambda p(x^2) + \mu q(x^2) \\
&= \lambda S_a(p(x)) + \mu S_a(q(x)).
\end{aligned}$$

S_c is linear because

$$\begin{aligned}
S_c(\lambda p(x) + \mu q(x)) &= S_c(\lambda(a_0 + a_1 x + a_2 x^2 + \cdots + a_d x^d) + \mu(b_0 + b_1 x + b_2 x^2 + \cdots + b_d x^d)) \\
&= S_c((\lambda a_0 + \mu b_0) + (\lambda a_1 + \mu b_1)x + (\lambda a_2 + \mu b_2)x^2 + \cdots + (\lambda a_d + \mu b_d)x^d) \\
&= x^2[(\lambda a_0 + \mu b_0) + (\lambda a_1 + \mu b_1)x + (\lambda a_2 + \mu b_2)x^2 + \cdots + (\lambda a_d + \mu b_d)x^d] \\
&= (\lambda a_0 + \mu b_0)x^2 + (\lambda a_1 + \mu b_1)x^3 + (\lambda a_2 + \mu b_2)x^4 + \cdots + (\lambda a_d + \mu b_d)x^{d+2} \\
&= \lambda x^2(a_0 + a_1 x + a_2 x^2 + \cdots + a_d x^d) + \mu x^2(b_0 + b_1 x + b_2 x^2 + \cdots + b_d x^d) \\
&= \lambda x^2 p(x) + \mu x^2 q(x) \\
&= \lambda S_c(p(x)) + \mu S_a(q(x)).
\end{aligned}$$

S_d is linear because

$$\begin{aligned}
S_d(\lambda p(x) + \mu q(x)) &= S_d(\lambda(a_0 + a_1 x + a_2 x^2 + \cdots + a_d x^d) + \mu(b_0 + b_1 x + b_2 x^2 + \cdots + b_d x^d)) \\
&= S_d((\lambda a_0 + \mu b_0) + (\lambda a_1 + \mu b_1)x + (\lambda a_2 + \mu b_2)x^2 + \cdots + (\lambda a_d + \mu b_d)x^d) \\
&= (\lambda a_0 + \mu b_0) + (\lambda a_1 + \mu b_1)(x+2) + (\lambda a_2 + \mu b_2)(x+2)^2 + \cdots + (\lambda a_d + \mu b_d)(x+2)^d \\
&= \lambda(a_0 + a_1(x+2) + a_2(x+2)^2 + \cdots + a_d(x+2)^d) \\
&\qquad + \mu(b_0 + b_1(x+2) + b_2(x+2)^2 + \cdots + b_d(x+2)^d) \\
&= \lambda p(x+2) + \mu q(x+2) \\
&= \lambda S_d(p(x)) + \mu S_d(q(x)).
\end{aligned}$$

S_b is not linear since

$$S_b(-x) = (-x)^2 = x^2 \neq -x^2 = -S_c(x).$$

[1] If, for example, $\deg(p) > \deg(q)$, then $d = \deg(p)$ and it is understood that $b_{\deg(q)+1} = \cdots = b_d = 0$.

5. We have shown that $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}$ is a vector space over $\mathbb{R}$ with addition and scalar multiplication defined by

$$x \oplus y = xy \qquad \text{and} \qquad \lambda \odot x = x^\lambda$$

for $x, y \in \mathbb{R}_+$, $\lambda \in \mathbb{R}$. Show that $\log_2 : \mathbb{R}_+ \to \mathbb{R}$ is a linear transformation. What about $\log_b : \mathbb{R}_+ \to \mathbb{R}$ for other bases $b > 2$?

SOLUTION. Let $\alpha, \beta \in \mathbb{R}$ and $x, y \in \mathbb{R}_+$. Then

$$\begin{aligned}
\log_2(\alpha \odot x \oplus \beta \odot y) &= \log_2(x^\alpha \oplus y^\beta) \\
&= \log_2(x^\alpha y^\beta) \\
&= \log_2(x^\alpha) + \log_2(y^\beta) \\
&= \alpha \log_2(x) + \beta \log_2(y)
\end{aligned}$$

by the basic properties of logarithms. Hence $\log_2 : \mathbb{R}_+ \to \mathbb{R}$ is a linear transformation. Again using a basic property of logarithms, namely,

$$\log_b(x) = \frac{\log_2(x)}{\log_2(b)},$$

we see that

$$\log_b = \gamma \log_2$$

where $\gamma = 1/\log_2(b) \in \mathbb{R}$. By a result in the lectures, any scalar multiple of a linear transformation must also be a linear transformation. So $\log_b : \mathbb{R}_+ \to \mathbb{R}$ is a linear transformation.