1. Let $W_1$ and $W_2$ be the following subspaces of $\mathbb{R}^4$.

$$W_1 = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}, \quad W_2 = \text{span}\left\{ \begin{bmatrix} 2 \\ 5 \\ -6 \\ -5 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -7 \\ 3 \end{bmatrix} \right\}.$$  

(a) Find a basis and state the dimension of $W_1 \cap W_2$.

**SOLUTION.** Let the three vectors spanning $W_1$ be $u_1, u_2, u_3$ and the two vectors spanning $W_2$ be $v_1, v_2$ respectively. If $w \in W_1 \cap W_2$, then there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ and $\beta_1, \beta_2 \in \mathbb{R}$ such that

$$\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = w = \beta_1 v_1 + \beta_2 v_2.$$  

In other words, we want to solve

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 2 \beta_1 + 2 \beta_2 \\ 5 \beta_1 + 2 \beta_2 \\ -6 \beta_1 - 7 \beta_2 \\ -5 \beta_1 + 3 \beta_2 \end{bmatrix}.$$  

Forming the augmented system to solve for $\alpha_1, \alpha_2, \alpha_3$ in terms of $\beta_1, \beta_2$ (alternatively, we could also solve for $\beta_1$ and $\beta_2$ in terms of $\alpha_1, \alpha_2, \alpha_3$), we get upon Gauss-Jordan elimination,

$$\begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 \beta_1 + 2 \beta_2 \\ 5 \beta_1 + 2 \beta_2 \\ -6 \beta_1 - 7 \beta_2 \\ -5 \beta_1 + 3 \beta_2 \end{bmatrix}.$$  

For this system to be consistent, we must have $\beta_2 = 0$. Hence

$$w = \beta_1 [2, 5, -6, -5]^T$$

for some $\beta_2 \in \mathbb{R}$. In other words,

$$W_1 \cap W_2 = \text{span}\{ [2, 5, -6, -5]^T \}$$

and so $\dim(W_1 \cap W_2) = 1$. As a sanity check, observe that $\beta_2 = 0$ gives

$$\alpha_1 = 3 \beta_1, \quad \alpha_2 = -\beta_1, \quad \alpha_3 = -2 \beta_1$$

and indeed $w$ can also be expressed as

$$w = 3 \beta_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \beta_1 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} - 2 \beta_1 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \in W_1.$$  

(b) Find a basis and state the dimension of $W_1 + W_2$.

**SOLUTION.** Observe that

$$\text{span}(S_1) + \text{span}(S_2) = \text{span}(S_1 \cup S_2).$$

So

$$W_1 + W_2 = \text{span}\{u_1, u_2, u_3, v_1, v_2\}.$$
We will apply the Throwing-Out Algorithm to get a basis. First we try to solve

\[
v_2 = \alpha u_1 + \beta u_2 + \gamma u_3 + \delta v_1.
\]  
(1.1)

Applying Gauss-Jordan Elimination, we get

\[
\begin{bmatrix}
1 & 3 & -1 & 2 & -1 \\
2 & 1 & 0 & 5 & 2 \\
-1 & 1 & 1 & -6 & -7 \\
-2 & 1 & -1 & -5 & 3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & -3 & 2 \\
0 & 1 & 0 & 1 & -2 \\
0 & 0 & 1 & 2 & -9 \\
0 & 0 & 0 & 0 & -6
\end{bmatrix}
\]

which shows that (1.1) has no solution — so we cannot discard \(v_2\). Next we try to solve

\[
v_1 = \alpha u_1 + \beta u_2 + \gamma u_3 + \delta v_2.
\]  
(1.2)

Applying Gauss-Jordan Elimination, we get

\[
\begin{bmatrix}
1 & 3 & -1 & 2 \\
2 & 1 & 2 & 5 \\
-1 & 1 & -7 & 1 \\
-2 & 1 & 3 & -1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & -2 \\
0 & 0 & 0 & 2
\end{bmatrix}
\]

which shows that (1.2) has a solution (given by \(\alpha = 3, \beta = -1, \gamma = -2, \delta = 0\)). Hence we may discard \(v_1\). Next we try to solve

\[
u_3 = \alpha u_1 + \beta u_2 + \gamma v_2.
\]  
(1.3)

Applying Gauss-Jordan Elimination, we get

\[
\begin{bmatrix}
1 & 3 & -1 & -1 \\
2 & 1 & 2 & 0 \\
-1 & 1 & -7 & 1 \\
-2 & 1 & 3 & -1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

which shows that (1.3) has no solution — so we cannot discard \(u_3\). Next we try to solve

\[
u_2 = \alpha u_1 + \beta u_3 + \gamma v_2.
\]  
(1.4)

Applying Gauss-Jordan Elimination, we get

\[
\begin{bmatrix}
1 & 1 & -1 & 1 \\
2 & 0 & 2 & 1 \\
-1 & 1 & -7 & 1 \\
-2 & 1 & 3 & -1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & 0 & -\frac{3}{2} \\
0 & 0 & 1 & -\frac{3}{2} \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

which shows that (1.4) has no solution — so we cannot discard \(u_2\). Lastly we try to solve

\[
u_1 = \alpha u_2 + \beta u_3 + \gamma v_2.
\]  
(1.5)

Applying Gauss-Jordan Elimination, we get

\[
\begin{bmatrix}
3 & -1 & -1 & 1 \\
1 & 0 & 2 & 2 \\
1 & 1 & -7 & -1 \\
1 & -1 & 3 & -2
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & -1 \\
0 & 1 & 0 & -\frac{3}{4} \\
0 & 0 & 1 & -\frac{3}{4} \\
0 & 0 & 0 & 11
\end{bmatrix}
\]

which shows that (1.5) has no solution — so we cannot discard \(u_1\). Hence by the Throwing-Out Algorithm the set \(\{u_1, u_2, u_3, v_2\}\) is a basis for \(W_1 + W_2\) and so \(\dim(W_1 + W_2) = 4\).

[Note: We will soon learn a more efficient method to determine a basis from a spanning set of vectors.]

2. Let \(F\) be a field. Let \(W_1\) and \(W_2\) be the following subspaces of \(F^n\).

\[
W_1 = \{(x_1, x_2, \ldots, x_n) \in F^n \mid x_1 + x_2 + \cdots + x_n = 0\},
\]

\[
W_2 = \{(x_1, x_2, \ldots, x_n) \in F^n \mid x_1 = x_2 = \cdots = x_n\}.
\]
(a) Find a basis and state the dimension of $W_1$.

**Solution.** Let $(x_1, x_2, \ldots, x_n) \in W_1$. Then

$$(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, -x_1 - x_2 - \cdots - x_{n-1}).$$

Note that

$$
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_{n-1} \\
  -x_1 - x_2 - \cdots - x_{n-1}
\end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 1 \end{bmatrix} + \cdots + x_{n-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}
$$

and so the vectors $u_1 = (1, 0, \ldots, 0, -1)$, $u_2 = (0, 1, \ldots, 0, -1)$, \ldots, $u_{n-1} = (0, 0, \ldots, 1, -1)$ span $W_1$. If

$$\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_{n-1} u_{n-1} = 0,$$

then $(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, -\alpha_1 - \alpha_2 - \cdots - \alpha_{n-1}) = (0, 0, \ldots, 0, 0)$ and so $\alpha_1 = \alpha_2 = \cdots = \alpha_{n-1} = 0$. Hence $u_1, \ldots, u_{n-1}$ are linearly independent. So $\{u_1, \ldots, u_{n-1}\}$ is a basis for $W_1$ and $\dim(W_1) = n - 1$.

(b) Find a basis and state the dimension of $W_2$.

**Solution.** Let $(x_1, x_2, \ldots, x_n) \in W_2$. Then

$$(x_1, x_2, \ldots, x_n) = (x_1, x_1, \ldots, x_1) = x_1(1, 1, \ldots, 1).$$

So the vector $u = (1, 1, \ldots, 1)$ spans $W_2$. If $\alpha u = 0$, then $\alpha = 0$ since $u \neq 0$. Hence $\{u\}$ is a basis for $W_2$ and $\dim(W_2) = 1$.

(c) Show that $\mathbb{R}^n = W_1 \oplus W_2$.

**Solution.** Let $w \in W_1 \cap W_2$. Then $w \in W_2$ implies $w = (\alpha, \alpha, \ldots, \alpha)$ but then $w \in W_1$ implies that $\alpha + \alpha + \cdots + \alpha = 0$, so $n\alpha = 0$, so $\alpha = 0$, and so $w = 0$. By Homework 4, Problem 4(a),

$$\dim(W_1 + W_2) + \dim(W_1 \cap W_2) = \dim(W_1) + \dim(W_2) = n - 1 + 1 = n.$$

Since $\dim(W_1 \cap W_2) = \dim(\{0\}) = 0$, we have that $\dim(W_1 + W_2) = n = \dim(\mathbb{R}^n)$. Since $W_1 + W_2$ is a subspace of $\mathbb{R}^n$, we get $W_1 + W_2 = \mathbb{R}^n$. [Note: This statement is not true in general. For example, if $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$, i.e. the field of two elements $(0, 1)$ under binary arithmetic, then $\mathbb{F}^2 \neq W_1 \oplus W_2$ since $W_1 \cap W_2 \neq \{0_{\mathbb{F}^2}\}$. Why? Because $(1, 1) \in W_1$ (recall that $1 + 1 = 0$ in $\mathbb{F}$) and $(1, 1) \in W_2$ but $0_{\mathbb{F}^2} = (0, 0) \neq (1, 1)$].

3. Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. Which of the following functions $T_i : \mathbb{R}^2 \to \mathbb{R}^2$ ($i = a, b, c, d$) are linear? Justify your answers.

(a) $T_a(x, y) = (\alpha x + \beta y, \gamma x + \delta y)$,

(b) $T_b(x, y) = (\alpha x^2 + \beta y^2, \gamma x^2 + \delta y^2)$,

(c) $T_c(x, y) = (\alpha x + \beta y, \gamma x^2 + \delta y^2)$,

(d) $T_d(x, y) = (\alpha x + \beta y, \gamma x + \delta y)$,

where $(x, y) \in \mathbb{R}^2$.

**Solution.** $T_a$ and $T_c$ are linear. Let $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ and $\lambda, \mu \in \mathbb{R}$. Then

$$
T_a(\lambda(x_1, y_1) + \mu(x_2, y_2)) = T_a(\lambda x_1 + \mu x_2, \lambda y_1 + \mu y_2) = (\alpha(\lambda x_1 + \mu x_2) + \beta(\lambda y_1 + \mu y_2), \gamma(\lambda x_1 + \mu x_2) + \delta(\lambda y_1 + \mu y_2)) = (\alpha\lambda x_1 + \alpha\mu x_2 + \beta\lambda y_1 + \beta\mu y_2, \gamma\lambda x_1 + \gamma\mu x_2 + \delta\lambda y_1 + \delta\mu y_2)
$$

$$
= (\lambda(\alpha x_1 + \beta y_1), \gamma(\lambda x_1 + \delta y_1) + \mu(\alpha x_2 + \beta y_2, \gamma x_2 + \delta y_2)) = \lambda(T_a(x_1, y_1) + \mu T_a(x_2, y_2)).
$$
$T_c$ is just $T_a$ with $\alpha, \beta, \gamma, \delta$ (which can be any constants in $\mathbb{R}$) being $\alpha^2, \beta^2, \gamma^2, \delta^2$. $T_b$ is not linear unless $\alpha = \beta = \gamma = \delta = 0$. Say, suppose $\gamma \neq 0$, then

$$T_b(-1, 0) = T_b(-1, 0) = (\alpha, \gamma) \neq (-\alpha, -\gamma) = -T_b(1, 0)$$

since $\gamma \neq -\gamma$. $T_d$ is not linear unless $\beta = \gamma = 0$ since

$$T_d(0, 0) = (\beta, \gamma) \neq (0, 0)$$

if $\beta \neq 0$ or $\gamma \neq 0$.

4. Let $\mathbb{P}$ be the vector space of polynomials with coefficients in $\mathbb{R}$. Which of the following functions $S_i : \mathbb{P} \to \mathbb{P}$ ($i = a, b, c, d$) are linear? Justify your answers.

(a) $S_a(p(x)) = p(x^2)$,
(b) $S_b(p(x)) = [p(x)]^2$,
(c) $S_c(p(x)) = x^2p(x)$,
(d) $S_d(p(x)) = p(x + 2)$,

where $p(x) \in \mathbb{P}$.

**Solution.** $S_a, S_c,$ and $S_d$ are linear. Let $p(x), q(x) \in \mathbb{P}$ and $\lambda, \mu \in \mathbb{R}$. Let

$$d = \max\{\deg(p(x)), \deg(q(x))\}.$$ 

Then we may write

$$p(x) = a_0 + a_1x + \cdots + a_dx^d \quad \text{and} \quad q(x) = b_0 + b_1x + \cdots + b_dx^d.$$ 

$S_a$ is linear because

$$S_a(\lambda p(x) + \mu q(x)) = S_a(\lambda(a_0 + a_1x + a_2x^2 + \cdots + a_dx^d) + \mu(b_0 + b_1x + b_2x^2 + \cdots + b_dx^d))$$

$$= S_a(\lambda(a_0 + a_1x + a_2x^2 + \cdots + a_dx^d) + \mu(b_0 + b_1x + b_2x^2 + \cdots + b_dx^d))$$

$S_c$ is linear because

$$S_c(\lambda p(x) + \mu q(x)) = S_c(\lambda(a_0 + a_1x + a_2x^2 + \cdots + a_dx^d) + \mu(b_0 + b_1x + b_2x^2 + \cdots + b_dx^d))$$

$$= S_c(\lambda(a_0 + a_1x + a_2x^2 + \cdots + a_dx^d) + \mu(b_0 + b_1x + b_2x^2 + \cdots + b_dx^d))$$

$S_d$ is linear because

$$S_d(\lambda p(x) + \mu q(x)) = S_d(\lambda(a_0 + a_1x + a_2x^2 + \cdots + a_dx^d) + \mu(b_0 + b_1x + b_2x^2 + \cdots + b_dx^d))$$

$$= S_d(\lambda(a_0 + a_1x + a_2x^2 + \cdots + a_dx^d) + \mu(b_0 + b_1x + b_2x^2 + \cdots + b_dx^d))$$

$S_b$ is not linear since

$$S_b(-x) = (-x)^2 = x^2 \neq -x^2 = -S_c(x).$$

---

1If, for example, $\deg(p) > \deg(q)$, then $d = \deg(p)$ and it is understood that $b_{\deg(q)+1} = \cdots = b_d = 0.$
5. We have shown that \( R_+ = \{ x \in \mathbb{R} \mid x > 0 \} \) is a vector space over \( \mathbb{R} \) with addition and scalar multiplication defined by
\[
x \oplus y = xy \quad \text{and} \quad \lambda \circ x = x^\lambda
\]
for \( x, y \in R_+, \lambda \in \mathbb{R} \). Show that \( \log_2 : R_+ \rightarrow \mathbb{R} \) is a linear transformation. What about \( \log_b : R_+ \rightarrow \mathbb{R} \) for other bases \( b > 2 \)?

SOLUTION. Let \( \alpha, \beta \in \mathbb{R} \) and \( x, y \in R_+ \). Then
\[
\log_2 (\alpha \circ x \oplus \beta \circ y) = \log_2 (x^\alpha \oplus y^\beta)
= \log_2 (x^\alpha y^\beta)
= \log_2 (x^\alpha) + \log_2 (y^\beta)
= \alpha \log_2 (x) + \beta \log_2 (y)
\]
by the basic properties of logarithms. Hence \( \log_2 : R_+ \rightarrow \mathbb{R} \) is a linear transformation. Again using a basic property of logarithms, namely,
\[
\log_b (x) = \frac{\log_2 (x)}{\log_2 (b)},
\]
we see that
\[
\log_b = \gamma \log_2
\]
where \( \gamma = 1/\log_2 (b) \in \mathbb{R} \). By a result in the lectures, any scalar multiple of a linear transformation must also be a linear transformation. So \( \log_b : R_+ \rightarrow \mathbb{R} \) is a linear transformation.