MATH 110: LINEAR ALGEBRA
FALL 2007/08
PROBLEM SET 3

1. Let W, W_1, W_2 be subspaces of a vector space V. Recall from Problem Set 2 that

$$W \cap (W_1 + W_2) \neq (W \cap W_1) + (W \cap W_2)$$

in general. Show that the following statements are however true:

(a) $W \cap (W_1 + W_2) \supseteq (W \cap W_1) + (W \cap W_2)$.

Solution. Since $W_1 \subseteq W_1 + W_2$, so $W \cap W_1 \subseteq W \cap (W_1 + W_2)$. Likewise, since $W_2 \subseteq W_1 + W_2$, so $W \cap W_2 \subseteq W \cap (W_1 + W_2)$. Hence

$$(W \cap W_1) \cup (W \cap W_2) \subseteq W \cap (W_1 + W_2).$$

Now since W and $W_1 + W_2$ are both subspaces of V, $W \cap (W_1 + W_2)$ is also a subspace of V. Now by Problem **3**(c) in Problem Set 2, the smallest subspace containing $(W \cap W_1) \cup (W \cap W_2)$ is $(W \cap W_1) + (W \cap W_2)$, which implies that

$$(W \cap W_1) + (W \cap W_2) \subseteq W \cap (W_1 + W_2). \tag{1.1}$$

(b) $W \cap (W_1 + (W \cap W_2)) = (W \cap W_1) + (W \cap W_2)$.

Solution. Since

$$W \cap (W \cap W_2) = (W \cap W) \cap W_2 = W \cap W_2,$$

it follows from (a), with $W \cap W_2$ playing the role of W_2 in (1.1), that

$$(W \cap W_1) + (W \cap W_2) = (W \cap W_1) + (W \cap (W \cap W_2)) \subseteq W \cap (W_1 + (W \cap W_2)).$$

To show the reverse inclusion, let $\mathbf{w} \in W \cap (W_1 + (W \cap W_2))$. Then $\mathbf{w} \in W$ and $\mathbf{w} \in W_1 + (W \cap W_2)$. Hence $\mathbf{w} \in W$ and $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$ for some $\mathbf{w}_1 \in W_1$ and $\mathbf{w}_2 \in W \cap W_2$. Observe that $\mathbf{w}_2 \in W \cap W_2$ implies that $\mathbf{w}_2 \in W$; and noting that $\mathbf{w} \in W$, we get

$$\mathbf{w}_1 = \mathbf{w}_2 - \mathbf{w} \in W$$

since W is a subspace. Now $\mathbf{w}_1 \in W_1$ and $\mathbf{w}_1 \in W$ together implies that $\mathbf{w}_1 \in W \cap W_1$. It then follows that $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2 \in (W \cap W_1) + (W \cap W_2)$. Since this is true for arbitrary $\mathbf{w} \in W \cap (W_1 + (W \cap W_2))$, we conclude that

$$W \cap (W_1 + (W \cap W_2)) \subseteq (W \cap W_1) + (W \cap W_2).$$

2. Let V be a vector space and $\mathbf{u}_1, \dots, \mathbf{u}_n \in V$.

(a) Suppose $\mathbf{w} \in \operatorname{span}\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$. Show that

$$\operatorname{span}\{\mathbf{u}_1, \dots, \mathbf{u}_n, \mathbf{w}\} = \operatorname{span}\{\mathbf{u}_1, \dots, \mathbf{u}_n\}.$$

Solution. Let $\mathbf{v} \in \operatorname{span}\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$. Then

$$\mathbf{v} = \lambda_1 \mathbf{u}_1 + \cdots + \lambda_n \mathbf{u}_n$$

for some $\lambda_1, \dots, \lambda_n \in \mathbb{F}$. But since $0\mathbf{w} = \mathbf{0}$, we could write

$$\mathbf{v} = \lambda_1 \mathbf{u}_1 + \cdots + \lambda_n \mathbf{u}_n + 0\mathbf{w}$$

and so $\mathbf{v} \in \operatorname{span}\{\mathbf{u}_1, \dots, \mathbf{u}_n, \mathbf{w}\}$. For the reverse inclusion, let $\mathbf{v} \in \operatorname{span}\{\mathbf{u}_1, \dots, \mathbf{u}_n, \mathbf{w}\}$. Then

$$\mathbf{v} = \lambda_1 \mathbf{u}_1 + \cdots + \lambda_n \mathbf{u}_n + \lambda_{n+1} \mathbf{w} \tag{2.2}$$

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for some $\lambda_1, \ldots, \lambda_{n+1} \in \mathbb{F}$. Since $\mathbf{w} \in \operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$,

$$\mathbf{w} = \mu_1 \mathbf{u}_1 + \cdots + \mu_n \mathbf{u}_n \tag{2.3}$$

for some $\mu_1, \ldots, \mu_n \in \mathbb{F}$. Substituting (2.3) into (2.2) yields

$$\begin{aligned} \mathbf{v} &= \lambda_1 \mathbf{u}_1 + \cdots + \lambda_n \mathbf{u}_n + \lambda_{n+1}(\mu_1 \mathbf{u}_1 + \cdots + \mu_n \mathbf{u}_n) \\ &= (\lambda_1 + \lambda_{n+1}\mu_1)\mathbf{u}_1 + \cdots + (\lambda_n + \lambda_{n+1}\mu_1)\mathbf{u}_n \end{aligned}$$

and so $\mathbf{v} \in \operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$.

(b) Generalize Problem **2**(a) to more than one vector. Suppose $\mathbf{w}_1, \ldots, \mathbf{w}_m \in \operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$. Show that

$$\operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbf{w}_1, \ldots, \mathbf{w}_m\} = \operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}.$$

Solution. We do this by induction. If $m = 1$, then it is true by (a). Suppose this is true for $m - 1$, ie.

$$\operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbf{w}_1, \ldots, \mathbf{w}_{m-1}\} = \operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}.$$

Then since $\mathbf{w}_m \in \operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_n\} = \operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbf{w}_1, \ldots, \mathbf{w}_{m-1}\}$, we may apply (a) again to get

$$\begin{aligned} \operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbf{w}_1, \ldots, \mathbf{w}_{m-1}, \mathbf{w}_m\} &= \operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbf{w}_1, \ldots, \mathbf{w}_{m-1}\} \\ &= \operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}. \end{aligned}$$

Hence this is true for all $m \in \mathbb{N}$ by the priniciple of mathematical induction.

3. Let V be a vector space.

(a) Let W be a subspace of V. Show that $\operatorname{span}(W) = W$.

Solution. Since W is a subspace of V, W is closed under linear combinations, ie. for any $\mathbf{u}_1, \ldots, \mathbf{u}_n \in W$, any $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$, and any $n \in \mathbb{N}$,

$$\lambda_1 \mathbf{u}_1 + \cdots + \lambda_n \mathbf{u}_n \in W.$$

Since by its definition,

$$\operatorname{span}(W) = \{\lambda_1 \mathbf{u}_1 + \cdots + \lambda_n \mathbf{u}_n \in V \mid \mathbf{u}_1, \ldots, \mathbf{u}_n \in W;\ \lambda_1, \ldots, \lambda_n \in \mathbb{F};\ n \in \mathbb{N}\},$$

it follows that $\operatorname{span}(W) \subseteq W$. But clearly, $W \subseteq \operatorname{span}(W)$. Both inclusions then yield $\operatorname{span}(W) = W$.

(b) Let S be a subset of V. What is $\operatorname{span}(\operatorname{span}(S))$?

Solution. Since $\operatorname{span}(S)$ is a subspace, it follows from (a) that

$$\operatorname{span}(\operatorname{span}(S)) = \operatorname{span}(S).$$

(c) Let S and T be subsets of V. Show that if $S \subseteq T$, then $\operatorname{span}(S) \subseteq \operatorname{span}(T)$.

Solution. Let $\mathbf{v} \in \operatorname{span}(S)$. Then

$$\mathbf{v} = \lambda_1 \mathbf{u}_1 + \cdots + \lambda_n \mathbf{u}_n$$

for some $\mathbf{u}_1, \ldots, \mathbf{u}_n \in S$, $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$, and $n \in \mathbb{N}$. But $S \subseteq T$, so $\mathbf{u}_1, \ldots, \mathbf{u}_n \in T$. Thus $\mathbf{v} \in \operatorname{span}(T)$. Accordingly, $\operatorname{span}(S) \subseteq \operatorname{span}(T)$.

(d) Let S be a subset of V. What is $\operatorname{span}(S \cup \{\mathbf{0}\})$?

Solution. Since $S \subseteq S \cup \{\mathbf{0}\}$, it follows from (c) that $\operatorname{span}(S) \subseteq \operatorname{span}(S \cup \{\mathbf{0}\})$. Suppose $\mathbf{v} \in \operatorname{span}(S \cup \{\mathbf{0}\})$. Then

$$\mathbf{v} = \lambda_1 \mathbf{u}_1 + \cdots + \lambda_n \mathbf{u}_n + \lambda_{n+1}\mathbf{0}$$

for some $\mathbf{u}_1, \ldots, \mathbf{u}_n \in S$, $\lambda_1, \ldots, \lambda_{n+1} \in \mathbb{F}$, and $n \in \mathbb{N}$. But since $\lambda_{n+1}\mathbf{0} = \mathbf{0}$, we have

$$\mathbf{v} = \lambda_1 \mathbf{u}_1 + \cdots + \lambda_n \mathbf{u}_n$$

and so $\mathbf{v} \in \operatorname{span}(S)$. Thus $\operatorname{span}(S \cup \{\mathbf{0}\}) \subseteq \operatorname{span}(S)$. Both inclusions then yield

$$\operatorname{span}(S \cup \{\mathbf{0}\}) = \operatorname{span}(S).$$

Alternatively, just apply Problem **2**(a) with $\mathbf{w} = \mathbf{0}$.

4. Let W_1, W_2 be subspaces of a vector space V. Recall from Problem Set 2 that $W = W_1 \oplus W_2$ if and only if (i) $W = W_1 + W_2$ and (ii) $W_1 \cap W_2 = \{\mathbf{0}\}$ are both satisfied.

(a) Let $V = \mathbb{R}^3$. Let $W_1 = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}$ and $W_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\}$. Show that

$$\mathbb{R}^3 = W_1 \oplus W_2.$$

SOLUTION. Note that $(x, y, z) \in W_1 \cap W_2$ implies that $x = y = z$ and $x = 0$ and thus $x = 0, y = 0, z = 0$. Hence $W_1 \cap W_2 = \{(0, 0, 0)\}$. Now for any $(x, y, z) \in \mathbb{R}^3$, we may write

$$(x, y, z) = (x, x, x) + (0, y - x, z - x)$$

where $(x, x, x) \in W_1$ and $(0, y - x, z - x) \in W_2$. Hence $\mathbb{R}^3 = W_1 + W_2$. The two conditions $W_1 \cap W_2 = \{(0, 0, 0)\}$ and $\mathbb{R}^3 = W_1 + W_2$ imply that $\mathbb{R}^3 = W_1 \oplus W_2$.

(b) Let the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ be linearly independent. Show that

$$\operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{v}_1, \ldots, \mathbf{v}_n\} = \operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_m\} \oplus \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}.$$

SOLUTION. We know from Midterm 1, Problem **4**(b) that

$$\begin{aligned}\operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_m\} + \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} &= \operatorname{span}(\{\mathbf{u}_1, \ldots, \mathbf{u}_m\} \cup \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}) \\ &= \operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{v}_1, \ldots, \mathbf{v}_n\}.\end{aligned}$$

It remains to show that the sum is a direct sum (ie. we may write '$\oplus$' in place of '$+$'). In other words, we need the additional condition that

$$\operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_m\} \cap \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} = \{\mathbf{0}\}.$$

Suppose $\mathbf{w} \in \operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_m\} \cap \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Then we must have

$$\mathbf{w} = \lambda_1 \mathbf{u}_1 + \cdots + \lambda_m \mathbf{u}_m \quad \text{and} \quad \mathbf{w} = \mu_1 \mathbf{v}_1 + \cdots + \mu_n \mathbf{v}_n$$

for some $\lambda_1, \ldots, \lambda_m \in \mathbb{F}$ and $\mu_1, \ldots, \mu_n \in \mathbb{F}$. So we get

$$\lambda_1 \mathbf{u}_1 + \cdots + \lambda_m \mathbf{u}_m = \mu_1 \mathbf{v}_1 + \cdots + \mu_n \mathbf{v}_n$$

and thus

$$\lambda_1 \mathbf{u}_1 + \cdots + \lambda_m \mathbf{u}_m - \mu_1 \mathbf{v}_1 - \cdots - \mu_n \mathbf{v}_n = \mathbf{0}.$$

Since $\mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent, this implies

$$\lambda_1 = \cdots = \lambda_m = \mu_1 = \cdots = \mu_n = 0$$

and thus $\mathbf{w} = 0\mathbf{u}_1 + \cdots + 0\mathbf{u}_m = \mathbf{0}$. Hence $\operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_m\} \cap \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} = \{\mathbf{0}\}$ as required.

5. Determine whether the following are linearly dependent.

(a) The vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^4$ where

$$\mathbf{a} = (1, 2, 3, 1), \quad \mathbf{b} = (3, -1, 2, 2), \quad \mathbf{c} = (1, -5, -4, 0).$$

SOLUTION. Similar to (c).

(b) The matrices $A, B, C \in \mathbb{R}^{2\times 2}$ where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -1 \\ 2 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -5 \\ -4 & 0 \end{bmatrix}.$$

SOLUTION. Similar to (c).

(c) The polynomials $a(x), b(x), c(x) \in \mathbb{P}_3$ where

$$a(x) = 1 + 2x + 3x^2 + x^3, \quad b(x) = 3 - x + 2x^2 + 2x^3, \quad c(x) = 1 - 5x - 4x^2.$$

SOLUTION. Suppose $\alpha, \beta, \gamma \in \mathbb{R}$ are such that

$$\alpha a(x) + \beta b(x) + \gamma c(x) = \mathbf{0},$$

in other words,

$$\alpha(1+2x+3x^2+x^3)+\beta(3-x+2x^2+2x^3)+\gamma(1-5x-4x^2)=0+0x+0x^2+0x^3.$$

Then

$$(\alpha+3\beta+\gamma)+(2\alpha-\beta-5\gamma)x+(3\alpha+2\beta-4\gamma)x^2+(\alpha+2\beta)x^3=0+0x+0x^2+0x^3.$$

Equating corresponding coefficients gives the system

$$\begin{aligned}\alpha+3\beta+\gamma&=0,\\ 2\alpha-\beta-5\gamma&=0,\\ 3\alpha+2\beta-4\gamma&=0,\\ \alpha+2\beta&=0.\end{aligned}$$

Applying Gaussian Elimination, we may reduce this to

$$\begin{aligned}\alpha+3\beta+\gamma&=0,\\ \beta+\gamma&=0.\end{aligned}$$

So this solution to system involves a free parameter and hence there exists nonzero solutions. For example, $\alpha=2,\beta=-1,\gamma=1$, which gives

$$2a(x)-b(x)+c(x)=\mathbf{0}.$$

So the polynomials $a(x),b(x),c(x)$ are linearly dependent.

(d) The vectors $\mathbf{p},\mathbf{q}\in\mathbb{R}^6$ where

$$\mathbf{p}=(1,2,-3,4,-5,6),\quad \mathbf{q}=(6,-5,4,-3,2,1).$$

Solution. Similar to (e).

(e) The matrices $P,Q\in\mathbb{R}^{2\times 3}$ where

$$P=\begin{bmatrix}1&2&-3\\6&-5&4\end{bmatrix},\quad Q=\begin{bmatrix}6&-5&4\\1&2&-3\end{bmatrix}.$$

Solution. Suppose $\lambda,\mu\in\mathbb{R}$ are such that

$$\lambda P+\mu Q=\mathbf{0},$$

in other words,

$$\lambda\begin{bmatrix}1&2&-3\\6&-5&4\end{bmatrix}+\mu\begin{bmatrix}6&-5&4\\1&2&-3\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}.$$

Then

$$\begin{bmatrix}\lambda+6\mu&2\lambda-5\mu&-3\lambda+4\mu\\6\lambda+\mu&-5\lambda+2\mu&4\lambda-3\mu\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}.$$

Equating corresponding entries gives the system

$$\begin{aligned}\lambda+6\mu&=0,\\ 2\lambda-5\mu&=0,\\ -3\lambda+4\mu&=0,\\ 6\lambda+\mu&=0,\\ -5\lambda+2\mu&=0,\\ 4\lambda-3\mu&=0.\end{aligned}$$

This has the unique solution $\lambda=0$, $\mu=0$. Hence P and Q are linearly independent.

(f) The polynomials $p(x),q(x)\in\mathbb{P}_5$ where

$$a(x)=1+2x-3x^2+4x^3-5x^4+6x^5,\quad b(x)=6-5x+4x^2-3x^3+2x^4+x^5.$$

Solution. Similar to (e).

6. Let V be a vector space and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$.

(a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent, show that $\mathbf{u}+\mathbf{v}$, $\mathbf{u}-\mathbf{v}$, and $\mathbf{u}-2\mathbf{v}+\mathbf{w}$ are also linearly independent.

SOLUTION. Suppose

$$\mu_1(\mathbf{u}+\mathbf{v}) + \mu_2(\mathbf{u}-\mathbf{v}) + \mu_3(\mathbf{u}-2\mathbf{v}+\mathbf{w}) = \mathbf{0}$$

for some $a, b, c \in \mathbb{F}$. Therefore,

$$(\mu_1+\mu_2+\mu_3)\mathbf{u} + (\mu_1-\mu_2-2\mu_3)\mathbf{v} + \mu_3\mathbf{w} = \mathbf{0}.$$

Since $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent, we must have

$$\begin{aligned} \mu_1+\mu_2+\mu_3 &= 0, \\ \mu_1-\mu_2-2\mu_3 &= 0, \\ \mu_3 &= 0. \end{aligned}$$

The only solution to the above system is $\mu_1 = 0, \mu_2 = 0, \mu_3 = 0$. Hence $\mathbf{u}+\mathbf{v}$, $\mathbf{u}-\mathbf{v}$, and $\mathbf{u}-2\mathbf{v}+\mathbf{w}$ are linearly independent.

(b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent and they span V, show that $\mathbf{u}+\mathbf{v}$, $\mathbf{u}-\mathbf{v}$, and $\mathbf{u}-2\mathbf{v}+\mathbf{w}$ also span V.

SOLUTION. Let $\mathbf{x} \in V$. Then since $\mathbf{u}, \mathbf{v}, \mathbf{w}$ span V, there exist $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{F}$ such that

$$\mathbf{x} = \lambda_1\mathbf{u} + \lambda_2\mathbf{v} + \lambda_3\mathbf{w}.$$

We claim that there also exist $\mu_1, \mu_2, \mu_3 \in \mathbb{F}$ such that

$$\mathbf{x} = \mu_1(\mathbf{u}+\mathbf{v}) + \mu_2(\mathbf{u}-\mathbf{v}) + \mu_3(\mathbf{u}-2\mathbf{v}+\mathbf{w}).$$

In other words, the equation

$$\mu_1(\mathbf{u}+\mathbf{v}) + \mu_2(\mathbf{u}-\mathbf{v}) + \mu_3(\mathbf{u}-2\mathbf{v}+\mathbf{w}) = \lambda_1\mathbf{u} + \lambda_2\mathbf{v} + \lambda_3\mathbf{w} \tag{6.4}$$

has a solution for μ_1, μ_2, μ_3 in terms of $\lambda_1, \lambda_2, \lambda_3$. Now, (6.4) may be rewritten as

$$(\mu_1+\mu_2+\mu_3)\mathbf{u} + (\mu_1-\mu_2-2\mu_3)\mathbf{v} + \mu_3\mathbf{w} = \lambda_1\mathbf{u} + \lambda_2\mathbf{v} + \lambda_3\mathbf{w},$$

$$(\mu_1+\mu_2+\mu_3-\lambda_1)\mathbf{u} + (\mu_1-\mu_2-2\mu_3-\lambda_2)\mathbf{v} + (\mu_3-\lambda_3)\mathbf{w} = \mathbf{0}.$$

Since $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent, we must have

$$\begin{aligned} \mu_1+\mu_2+\mu_3 &= \lambda_1, \\ \mu_1-\mu_2-2\mu_3 &= \lambda_2, \\ \mu_3 &= \lambda_3, \end{aligned}$$

which can be solved to give

$$\mu_1 = (\lambda_1+\lambda_2+\lambda_3)/2, \quad \mu_2 = (\lambda_1-\lambda_2-3\lambda_3)/2, \quad \mu_3 = \lambda_3.$$

Hence $\mathbf{x} = \lambda_1\mathbf{u} + \lambda_2\mathbf{v} + \lambda_3\mathbf{w} \in V$ can be rewritten as

$$\mathbf{x} = \frac{1}{2}(\lambda_1+\lambda_2+\lambda_3)(\mathbf{u}+\mathbf{v}) + \frac{1}{2}(\lambda_1-\lambda_2-3\lambda_3)(\mathbf{u}-\mathbf{v}) + \lambda_3(\mathbf{u}-2\mathbf{v}+\mathbf{w}).$$

Since $\mathbf{x} \in V$ is arbitrary, it follows that $\mathbf{u}+\mathbf{v}$, $\mathbf{u}-\mathbf{v}$, and $\mathbf{u}-2\mathbf{v}+\mathbf{w}$ span V.