1. Let $W, W_1, W_2$ be subspaces of a vector space $V$. Recall from Problem Set 2 that

$$W \cap (W_1 + W_2) \neq (W \cap W_1) + (W \cap W_2)$$

in general. Show that the following statements are however true:

(a) $W \cap (W_1 + W_2) \supseteq (W \cap W_1) + (W \cap W_2)$.

**Solution.** Since $W_1 \subseteq W_1 + W_2$, so $W \cap W_1 \subseteq W \cap (W_1 + W_2)$. Likewise, since $W_2 \subseteq W_1 + W_2$, so $W \cap W_2 \subseteq W \cap (W_1 + W_2)$. Hence

$$(W \cap W_1) \cap (W \cap W_2) \subseteq W \cap (W_1 + W_2).$$

Now since $W$ and $W_1 + W_2$ are both subspaces of $V$, $W \cap (W_1 + W_2)$ is also a subspace of $V$. Now by Problem 3(c) in Problem Set 2, the smallest subspace containing $(W \cap W_1) \cap (W \cap W_2)$ is $(W \cap W_1) + (W \cap W_2)$, which implies that

$$(W \cap W_1) + (W \cap W_2) \subseteq W \cap (W_1 + W_2). \quad (1.1)$$

(b) $W \cap (W_1 + (W \cap W_2)) = (W \cap W_1) + (W \cap W_2)$.

**Solution.** Since

$$W \cap (W \cap W_2) = (W \cap W) \cap W_2 = W \cap W_2,$$

it follows from (a), with $W \cap W_2$ playing the role of $W_2$ in (1.1), that

$$(W \cap W_1) + (W \cap W_2) = (W \cap W_1) + (W \cap (W \cap W_2)) \subseteq W \cap (W_1 + (W \cap W_2)).$$

To show the reverse inclusion, let $w \in W \cap (W_1 + (W \cap W_2))$. Then $w \in W$ and $w \in W_1 + (W \cap W_2)$. Hence $w \in W$ and $w = w_1 + w_2$ for some $w_1 \in W_1$ and $w_2 \in W \cap W_2$. Observe that $w_2 \in W \cap W_2$ implies that $w_2 \in W$; and noting that $w \in W$, we get

$$w_1 = w_2 - w \in W$$

since $W$ is a subspace. Now $w_1 \in W_1$ and $w_1 \in W$ together implies that $w_1 \in W \cap W_1$. It then follows that $w = w_1 + w_2 \in (W \cap W_1) + (W \cap W_2)$. Since this is true for arbitrary $w \in W \cap (W_1 + (W \cap W_2))$, we conclude that

$$W \cap (W_1 + (W \cap W_2)) \subseteq (W \cap W_1) + (W \cap W_2).$$

2. Let $V$ be a vector space and $u_1, \ldots, u_n \in V$.

(a) Suppose $w \in \text{span}\{u_1, \ldots, u_n\}$. Show that

$$\text{span}\{u_1, \ldots, u_n, w\} = \text{span}\{u_1, \ldots, u_n\}.$$

**Solution.** Let $v \in \text{span}\{u_1, \ldots, u_n\}$. Then

$$v = \lambda_1 u_1 + \cdots + \lambda_n u_n$$

for some $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$. But since $0w = 0$, we could write

$$v = \lambda_1 u_1 + \cdots + \lambda_n u_n + 0w$$

and so $v \in \text{span}\{u_1, \ldots, u_n, w\}$. For the reverse inclusion, let $v \in \text{span}\{u_1, \ldots, u_n, w\}$. Then

$$v = \lambda_1 u_1 + \cdots + \lambda_n u_n + \lambda_{n+1} w \quad (2.2)$$
for some \( \lambda_1, \ldots, \lambda_{n+1} \in \mathbb{F} \). Since \( w \in \text{span}\{u_1, \ldots, u_n\} \),
\[
    w = \mu_1 u_1 + \cdots + \mu_n u_n
\]  
(2.3)
for some \( \mu_1, \ldots, \mu_n \in \mathbb{F} \). Substituting (2.3) into (2.2) yields
\[
    v = \lambda_1 u_1 + \cdots + \lambda_n u_n + \lambda_{n+1}(\mu_1 u_1 + \cdots + \mu_n u_n)
\]
and so \( v \in \text{span}\{u_1, \ldots, u_n\} \).

(b) Generalize Problem 2(a) to more than one vector. Suppose \( w_1, \ldots, w_m \in \text{span}\{u_1, \ldots, u_n\} \). Show that
\[
    \text{span}\{u_1, \ldots, u_n, w_1, \ldots, w_m\} = \text{span}\{u_1, \ldots, u_n\}.
\]

**Solution.** We do this by induction. If \( m = 1 \), then it is true by (a). Suppose this is true for \( m - 1 \), ie.
\[
    \text{span}\{u_1, \ldots, u_n, w_1, \ldots, w_{m-1}\} = \text{span}\{u_1, \ldots, u_n\}.
\]
Then since \( w_m \in \text{span}\{u_1, \ldots, u_n\} = \text{span}\{u_1, \ldots, u_n, w_1, \ldots, w_{m-1}\} \), we may apply (a) again to get
\[
    \text{span}\{u_1, \ldots, u_n, w_1, \ldots, w_{m-1}, w_m\} = \text{span}\{u_1, \ldots, u_n, w_1, \ldots, w_{m-1}\}
\]
\[
\substack{\text{= span}\{u_1, \ldots, u_n\}.}.
\]
Hence this is true for all \( m \in \mathbb{N} \) by the priniciple of mathematical induction.

3. Let \( V \) be a vector space.
(a) Let \( W \) be a subspace of \( V \). Show that \( \text{span}(W) = W \).

**Solution.** Since \( W \) is a subspace of \( V \), \( W \) is closed under linear combinations, ie. for any \( u_1, \ldots, u_n \in W \), any \( \lambda_1, \ldots, \lambda_n \in \mathbb{F} \), and any \( n \in \mathbb{N} \),
\[
    \lambda_1 u_1 + \cdots + \lambda_n u_n \in W.
\]
Since by its definition,
\[
    \text{span}(W) = \{\lambda_1 u_1 + \cdots + \lambda_n u_n \in V \mid u_1, \ldots, u_n \in W; \lambda_1, \ldots, \lambda_n \in \mathbb{F}; n \in \mathbb{N}\},
\]

it follows that \( \text{span}(W) \subseteq W \). But clearly, \( W \subseteq \text{span}(W) \). Both inclusions then yield \( \text{span}(W) = W \).

(b) Let \( S \) be a subset of \( V \). What is \( \text{span}(\text{span}(S)) \)?

**Solution.** Since \( \text{span}(S) \) is a subspace, it follows from (a) that
\[
    \text{span}(\text{span}(S)) = \text{span}(S).
\]

(c) Let \( S \) and \( T \) be subsets of \( V \). Show that if \( S \subseteq T \), then \( \text{span}(S) \subseteq \text{span}(T) \).

**Solution.** Let \( v \in \text{span}(S) \). Then
\[
    v = \lambda_1 u_1 + \cdots + \lambda_n u_n
\]
for some \( u_1, \ldots, u_n \in S \), \( \lambda_1, \ldots, \lambda_n \in \mathbb{F} \), and \( n \in \mathbb{N} \). But \( S \subseteq T \), so \( u_1, \ldots, u_n \in T \). Thus \( v \in \text{span}(T) \). Accordingly, \( \text{span}(S) \subseteq \text{span}(T) \).

(d) Let \( S \) be a subset of \( V \). What is \( \text{span}(S \cup \{0\}) \)?

**Solution.** Since \( S \subseteq S \cup \{0\} \), it follows from (c) that \( \text{span}(S) \subseteq \text{span}(S \cup \{0\}) \). Suppose \( v \in \text{span}(S \cup \{0\}) \). Then
\[
    v = \lambda_1 u_1 + \cdots + \lambda_n u_n + \lambda_{n+1} 0
\]
for some \( u_1, \ldots, u_n \in S \), \( \lambda_1, \ldots, \lambda_{n+1} \in \mathbb{F} \), and \( n \in \mathbb{N} \). But since \( \lambda_{n+1} 0 = 0 \), we have
\[
    v = \lambda_1 u_1 + \cdots + \lambda_n u_n
\]
and so \( v \in \text{span}(S) \). Thus \( \text{span}(S \cup \{0\}) \subseteq \text{span}(S) \). Both inclusions then yield
\[
    \text{span}(S \cup \{0\}) = \text{span}(S).
\]
5. Determine whether the following are linearly dependent.

(a) The vectors \( \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^4 \) where
\[
\mathbf{a} = (1, 2, 3, 1), \quad \mathbf{b} = (3, -1, 2, 2), \quad \mathbf{c} = (1, -5, -4, 0).
\]

**Solution.** Similar to (c).

(b) The matrices \( A, B, C \in \mathbb{R}^{2 \times 2} \) where
\[
A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -1 \\ 2 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -5 \\ -4 & 0 \end{bmatrix}.
\]

**Solution.** Similar to (c).

(c) The polynomials \( a(x), b(x), c(x) \in \mathbb{P}_3 \) where
\[
a(x) = 1 + 2x + 3x^2 + x^3, \quad b(x) = 3 - x + 2x^2 + 2x^3, \quad c(x) = 1 - 5x - 4x^2.
\]

**Solution.** Suppose \( \alpha, \beta, \gamma \in \mathbb{R} \) are such that
\[
\alpha a(x) + \beta b(x) + \gamma c(x) = 0,
\]
in other words,
\[ \alpha(1 + 2x + 3x^2 + x^3) + \beta(3 - x + 2x^2 + 2x^3) + \gamma(1 - 5x - 4x^2) = 0 + 0x + 0x^2 + 0x^3. \]

Then
\[ (\alpha + 3\beta + \gamma) + (2\alpha - \beta - 5\gamma)x + (3\alpha + 2\beta - 4\gamma)x^2 + (\alpha + 2\beta)x^3 = 0 + 0x + 0x^2 + 0x^3. \]
Equating corresponding coefficients gives the system
\[ \alpha + 3\beta + \gamma = 0, \]
\[ 2\alpha - \beta - 5\gamma = 0, \]
\[ 3\alpha + 2\beta - 4\gamma = 0, \]
\[ \alpha + 2\beta = 0. \]

Applying Gaussian Elimination, we may reduce this to
\[ \alpha + 3\beta + \gamma = 0, \]
\[ \beta + \gamma = 0. \]

So this solution to system involves a free parameter and hence there exists nonzero solutions.
For example, \( \alpha = 2, \beta = -1, \gamma = 1 \), which gives
\[ 2a(x) - b(x) + c(x) = 0. \]
So the polynomials \( a(x), b(x), c(x) \) are linearly dependent.

(d) The vectors \( p, q \in \mathbb{R}^6 \) where
\[ p = (1, 2, -3, 4, -5, 6), \quad q = (6, -5, 4, -3, 2, 1). \]
**Solution.** Similar to (e).

(e) The matrices \( P, Q \in \mathbb{R}^{2 \times 3} \) where
\[ P = \begin{bmatrix} 1 & 2 & -3 \\ 6 & -5 & 4 \end{bmatrix}, \quad Q = \begin{bmatrix} 6 & -5 & 4 \\ 1 & 2 & -3 \end{bmatrix}. \]
**Solution.** Suppose \( \lambda, \mu \in \mathbb{R} \) are such that
\[ \lambda P + \mu Q = 0, \]
in other words,
\[ \lambda \begin{bmatrix} 1 & 2 & -3 \\ 6 & -5 & 4 \end{bmatrix} + \mu \begin{bmatrix} 6 & -5 & 4 \\ 1 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \]
Then
\[ \begin{bmatrix} \lambda + 6\mu & 2\lambda - 5\mu & -3\lambda + 4\mu \\ 6\lambda + \mu & -5\lambda + 2\mu & 4\lambda - 3\mu \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \]
Equating corresponding entries gives the system
\[ \lambda + 6\mu = 0, \]
\[ 2\lambda - 5\mu = 0, \]
\[ -3\lambda + 4\mu = 0, \]
\[ 6\lambda + \mu = 0, \]
\[ -5\lambda + 2\mu = 0, \]
\[ 4\lambda - 3\mu = 0. \]
This has the unique solution \( \lambda = 0, \mu = 0 \). Hence \( P \) and \( Q \) are linearly independent.

(f) The polynomials \( p(x), q(x) \in \mathbb{P}_5 \) where
\[ a(x) = 1 + 2x - 3x^2 + 4x^3 - 5x^4 + 6x^5, \quad b(x) = 6 - 5x + 4x^2 - 3x^3 + 2x^4 + x^5. \]
**Solution.** Similar to (e).
6. Let $V$ be a vector space and $u, v, w \in V$.

(a) If $u, v, w$ are linearly independent, show that $u + v, u - v,$ and $u - 2v + w$ are also linearly independent.

**SOLUTION.** Suppose 

$$\mu_1(u + v) + \mu_2(u - v) + \mu_3(u - 2v + w) = 0$$

for some $a, b, c \in \mathbb{F}$. Therefore,

$$(\mu_1 + \mu_2 + \mu_3)u + (\mu_1 - \mu_2 - 2\mu_3)v + \mu_3w = 0.$$ 

Since $u, v, w$ are linearly independent, we must have

$$\mu_1 + \mu_2 + \mu_3 = 0,$$

$$\mu_1 - \mu_2 - 2\mu_3 = 0,$$

$$\mu_3 = 0.$$ 

The only solution to the above system is $\mu_1 = 0, \mu_2 = 0, \mu_3 = 0$. Hence $u + v, u - v,$ and $u - 2v + w$ are linearly independent.

(b) If $u, v, w$ are linearly independent and they span $V$, show that $u + v, u - v,$ and $u - 2v + w$ also span $V$.

**SOLUTION.** Let $x \in V$. Then since $u, v, w$ span $V$, there exist $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{F}$ such that 

$$x = \lambda_1 u + \lambda_2 v + \lambda_3 w.$$ 

We claim that there also exist $\mu_1, \mu_2, \mu_3 \in \mathbb{F}$ such that 

$$x = \mu_1(u + v) + \mu_2(u - v) + \mu_3(u - 2v + w).$$

In other words, the equation

$$\mu_1(u + v) + \mu_2(u - v) + \mu_3(u - 2v + w) = \lambda_1 u + \lambda_2 v + \lambda_3 w$$

has a solution for $\mu_1, \mu_2, \mu_3$ in terms of $\lambda_1, \lambda_2, \lambda_3$. Now, (6.4) may be rewritten as

$$(\mu_1 + \mu_2 + \mu_3)u + (\mu_1 - \mu_2 - 2\mu_3)v + \mu_3w = \lambda_1 u + \lambda_2 v + \lambda_3 w,$$

$$(\mu_1 + \mu_2 + \mu_3 - \lambda_1)u + (\mu_1 - \mu_2 - 2\mu_3 - \lambda_2)v + (\mu_3 - \lambda_3)w = 0.$$ 

Since $u, v, w$ are linearly independent, we must have

$$\mu_1 + \mu_2 + \mu_3 = \lambda_1,$$

$$\mu_1 - \mu_2 - 2\mu_3 = \lambda_2,$$

$$\mu_3 = \lambda_3,$$

which can be solved to give 

$$\mu_1 = (\lambda_1 + \lambda_2 + \lambda_3)/2, \quad \mu_2 = (\lambda_1 - \lambda_2 - 3\lambda_3)/2, \quad \mu_3 = \lambda_3.$$ 

Hence $x = \lambda_1 u + \lambda_2 v + \lambda_3 w \in V$ can be rewritten as 

$$x = \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3)(u + v) + \frac{1}{2}(\lambda_1 - \lambda_2 - 3\lambda_3)(u - v) + \lambda_3(u - 2v + w).$$

Since $x \in V$ is arbitrary, it follows that $u + v, u - v,$ and $u - 2v + w$ span $V$. 