MATH 110: LINEAR ALGEBRA
FALL 2007/08
PROBLEM SET 10

Elements in $\mathbb{R}^n$ will always be written as column vectors. The inner product on $\mathbb{R}^n$ will be denoted by $\langle \cdot, \cdot \rangle$ and its induced norm will be denoted $\| \cdot \|_2$. I will denote the identity matrix. $A \in \mathbb{R}^{m \times n}$ is called *full-rank* if $\operatorname{rank}(A) = \min\{m, n\}$. $M \in \mathbb{R}^{n \times n}$ is called *nonsymmetric* if $M^\top \neq M$.

1. A square matrix $M \in \mathbb{R}^{n \times n}$ is called *positive semidefinite* if

$$\mathbf{x}^\top M \mathbf{x} \geq 0$$

for all $\mathbf{x} \in \mathbb{R}^n$. $M \in \mathbb{R}^{n \times n}$ is called *positive definite* if (i) M is positive semidefinite; and (ii) $\mathbf{x}^\top M \mathbf{x} = 0$ only if $\mathbf{x} = \mathbf{0}$.

(a) Show that every positive definite matrix is nonsingular (ie. invertible).
(b) Show that if M is positive semidefinite and $\lambda \in \mathbb{R}$ is an eigenvalue of M, then $\lambda \geq 0$.
(c) Show that if M is positive definite and $\lambda \in \mathbb{R}$ is an eigenvalue of M, then $\lambda > 0$.
(d) Is it possible for a nonsymmetric matrix M to be positive semidefinite?
(e) Is it possible for a nonsymmetric matrix M to be positive definite?
(f) Let $A \in \mathbb{R}^{m \times n}$. Show that $A^\top A$ and $AA^\top$ are positive semidefinite matrices. Hence deduce that singular values are always nonnegative.
(g) Let $A \in \mathbb{R}^{m \times n}$ be full-rank. Show that either $A^\top A$ or $AA^\top$ is a positive definite matrix.

2. Let $\mathbf{u} \in \mathbb{R}^n$, $\mathbf{u} \neq \mathbf{0}$. Consider the *Householder* matrix $H_{\mathbf{u}} \in \mathbb{R}^{n \times n}$ defined by

$$H_{\mathbf{u}} = I - \frac{2\mathbf{u}\mathbf{u}^\top}{\|\mathbf{u}\|_2^2}.$$

(*Note*: $\mathbf{u}$ is a column vector, ie. an $n \times 1$ matrix, so $\mathbf{u}^\top$ is a row vector, ie. an $1 \times n$ matrix, and so $\mathbf{u}\mathbf{u}^\top$ is an $n \times n$ matrix).

(a) Show that $H_{\mathbf{u}}$ is both symmetric and orthogonal.
(b) Show that for any $\alpha \in \mathbb{R}$, $\alpha \neq 0$,

$$H_{\alpha\mathbf{u}} = H_{\mathbf{u}}.$$

In other words, $H_{\mathbf{u}}$ only depends on the 'direction' of $\mathbf{u}$ and not on its 'magnitude'.
(c) In general, given a matrix $M \in \mathbb{R}^{n \times n}$ and a vector $\mathbf{x} \in \mathbb{R}^n$, computing the matrix-vector product $M\mathbf{x}$ requires n inner products — one for each row of M with $\mathbf{x}$. Show that $H_{\mathbf{u}}\mathbf{x}$ can be computed using only two inner products.
(d) Given $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ where $\mathbf{a} \neq \mathbf{b}$ and $\|\mathbf{a}\|_2 = \|\mathbf{b}\|_2$. Find $\mathbf{u} \in \mathbb{R}^n$, $\mathbf{u} \neq \mathbf{0}$ such that

$$H_{\mathbf{u}}\mathbf{a} = \mathbf{b}.$$

(e) Show that $\mathbf{u}$ is an eigenvector of $H_{\mathbf{u}}$. What is the corresponding eigenvalue?
(f) Recall the definition of orthogonal complement in Problem Set **9**. Let $U = \operatorname{span}\{\mathbf{u}\}$. Show that every $\mathbf{v} \in U^\perp$ is an eigenvector of $H_{\mathbf{u}}$. What are the corresponding eigenvalues? What is $\dim(U^\perp)$?
(g) Using (d) and (e), find an orthogonal diagonalization of $H_{\mathbf{u}}$, ie. find an orthogonal matrix Q and a diagonal matrix Λ such that

$$H_{\mathbf{u}} = Q\Lambda Q^\top.$$

(*Hint*: you will need to use the Gram-Schmidt algorithm).

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3. (a) Let $Q \in \mathbb{R}^{n\times n}$ be an orthogonal matrix. Suppose $\lambda \in \mathbb{R}$ is an eigenvalue of Q. Show that either $\lambda = 1$ or $\lambda = -1$.

(b) Let $R \in \mathbb{R}^{n\times n}$ be an upper triangular matrix. What are the eigenvalues of R? Prove your claim.

(c) What is wrong with the following argument? Suppose we want to find the eigenvalues of $A \in \mathbb{R}^{n\times n}$ and we have found the QR-decomposition of A (say, via the Gram-Schmidt algorithm) to be

$$A = QR.$$

Now, if λ is an eigenvalue of Q and μ is an eigenvalue of R, then

$$A\mathbf{v} = QR\mathbf{v} = Q(\mu\mathbf{v}) = \mu Q\mathbf{v} = \mu\lambda\mathbf{v}$$

and so we have found $\mu\lambda$ to be an eigenvalue of A.

Remark: while the argument in (c) is incorrect, the QR-decomposition may indeed be used to compute eigenvalues and eigenvectors as we have discussed in the lectures; in fact, the QR-algorithm for computing eigenvalues was named one of the Top Ten Algorithms of the 20th Century: `http://math.berkeley.edu/~lekheng/courses/110/top10`.