1. Which one of the following functions is continuous at 0?

\[ e(x) = \begin{cases} 
1 & \text{if } x \in \mathbb{Q}, \\
0 & \text{if } x \notin \mathbb{Q},
\end{cases} \quad f(x) = \begin{cases} 
x & \text{if } x \in \mathbb{Q}, \\
0 & \text{if } x \notin \mathbb{Q},
\end{cases} \quad g(x) = \begin{cases} 
|x| & \text{if } x \in \mathbb{Q}, \\
0 & \text{if } x \notin \mathbb{Q},
\end{cases} \quad h(x) = \begin{cases} 
q & \text{if } x = p/q, \gcd(p,q) = 1, q > 0, \\
0 & \text{if } x \notin \mathbb{Q},
\end{cases} \]

(A) \( e \) and \( f \) only \quad (B) \( e \) and \( h \) only \quad (C) \( f \) and \( g \) only \quad (D) \( g \) and \( h \) only

Answer: By Examples 3.13, 3.14, and 4.4, \( e \) is discontinuous at 0 while \( f \) is continuous at 0. Since the absolute value function \( |\cdot| \) is continuous at 0, \( g = f \circ |\cdot| \) is continuous at 0 by Theorem 4.3. By Exercise 3.16, \( h \) is unbounded at 0 and is therefore discontinuous at 0 by the Lemma on pp. 72. Hence the answer is (C).

2. \( f, g, h \) are all discontinuous at \( x_0 \). Which of the following could be continuous at \( x_0 \)?

(A) \( f \circ (g + h) \) \quad (B) \( f \cdot (g + h) \) \quad (C) \( f \circ (g \cdot h) \) \quad (D) \( f \circ (g \circ h) \) \quad (E) all of them

Answer: Consider \( x_0 = 0 \). Let \( g : \mathbb{R} \to \mathbb{R} \) be defined by

\[ g(x) = \begin{cases} 
1 & \text{if } x > 0, \\
0 & \text{if } x \leq 0,
\end{cases} \]

and let \( h = -g \). So \( g + h = 0 \) and \( g \circ h = 0 \). So for any \( f \),

\[ f \circ (g + h) = f \cdot (g + h) = f \circ (g \circ h) = 0 \]

and are continuous. Note that \( g \cdot h = h \). So if we pick \( f = g \), then \( f \circ (g \cdot h) = g \circ h = 0 \) and is also continuous. So the answer is (E).

3. \( f \) is continuous on \( \mathbb{R} \) and \([f(x)]^4 = x^4\) for all \( x \in \mathbb{Q} \). A possible value of \( f(\sqrt{2}) \) is:

(A) 4 \quad (B) 2 \quad (C) -2 \quad (D) -\sqrt{2} \quad (E) any value in \( \mathbb{R} \) is possible

Answer: By Exercise 4.13, \([f(x)]^4 = x^4\) for all \( x \in \mathbb{R} \). Note that \([f(x)]^4 = x^4\) implies that \([f(x)]^2 = x^2\). So by Exercise 4.17, the four possibilities of \( f \) are \(-x, x, -|x|, \text{or } |x|\). The only possible answer is (D).

4. What is the value of \( \lim_{n \to \infty} n \sin(2/n) \)?

(A) 0 \quad (B) 1 \quad (C) 2 \quad (D) \infty \quad (E) limit does not exist
**Answer:** By Exercise 3.29, 
\[
\lim_{x \to 0} \frac{\sin x}{x} = 1.
\]

By Theorem 3.6, if we choose \(x_n = \frac{2}{n}\) and note that \(x_n \to 0\), then 
\[
\lim_{n \to \infty} \frac{\sin x_n}{x_n} = 1.
\]

Now note that 
\[
\lim_{n \to \infty} n \sin \left( \frac{2}{n} \right) = 2 \lim_{n \to \infty} \frac{\sin(2/n)}{2/n} = 2.
\]

Alternatively apply Exercise 3.30(b). So the answer is (C).

**5.** What is the value of the following? 
\[
\lim_{x \to 0} \frac{\tan x + \sqrt{4 + x} - 2}{x}
\]

(A) 1  (B) 1/2  (C) 1/4  (D) 0  (E) limit does not exist

**Answer:** Note that by Theorem 3.4(b) and Exercise 3.29,
\[
\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\sin x}{x} \times \lim_{x \to 0} \frac{1}{\cos x} = 1.
\]

Note that as in Exercise 3.23(a),
\[
\lim_{x \to 0} \frac{\sqrt{4 + x} - 2}{x} = \lim_{x \to 0} \frac{(\sqrt{4 + x} - 2)(\sqrt{4 + x} + 2)}{x(\sqrt{4 + x} + 2)} = \lim_{x \to 0} \frac{4 + x - 4}{x(\sqrt{4 + x} + 2)} = \lim_{x \to 0} \frac{1}{\sqrt{4 + x} + 2} = \frac{1}{4}.
\]

Alternatively, use l’hôpital’s rule. Hence by Theorem 3.4(a),
\[
\lim_{x \to 0} \frac{\tan x + \sqrt{4 + x} - 2}{x} = 1 + \frac{1}{4} = \frac{5}{4}.
\]

None of the given answers are correct.

**6.** What can you say\(^1\) about the following statement: “For any \(\varepsilon \in (0, 2)\), there exists \(\delta > 0\) such that \(|f(x) - 3| < 4\varepsilon\) whenever \(0 < |x - 5| < \delta\).”

(A) This is a necessary but not sufficient condition for \(\lim_{x \to 5} f(x) = 3\).
(B) This is a sufficient but not necessary condition for \(\lim_{x \to 5} f(x) = 3\).
(C) This is both a necessary and a sufficient condition for \(\lim_{x \to 5} f(x) = 3\).
(D) This is neither a necessary nor a sufficient condition for \(\lim_{x \to 5} f(x) = 3\).
(E) This condition contradicts \(\lim_{x \to 5} f(x) = 3\).

**Answer:** Let \(P_1\) be the statement “For any \(\varepsilon \in (0, 2)\), there exists \(\delta > 0\) such that \(|f(x) - 3| < 4\varepsilon\) whenever \(0 < |x - 5| < \delta\)” and \(P_2\) be the statement “For any \(\varepsilon > 0\), there exists \(\delta > 0\) such that \(|f(x) - 3| < \varepsilon\) whenever \(0 < |x - 5| < \delta\).” We claim that \(P_1 \iff P_2\), i.e. the answer is (C). It is clear that \(P_2 \implies P_1\). Now assuming \(P_1\). Given any \(\varepsilon > 0\), if \(\varepsilon < 8\), then let \(\varepsilon' = \varepsilon/4 \in (0, 2)\) and so there exists \(\delta > 0\) such that \(|f(x) - 3| < 4\varepsilon' = \varepsilon\) whenever \(0 < |x - 5| < \delta\); if \(\varepsilon \geq 8\), then let \(\varepsilon' = 1 \in (0, 2)\) (any other choice of \(\varepsilon' \in (0, 2)\) would work too) and so again so there exists

\(^1\)Recall if \(P \implies Q\), we say ‘\(P\) is a sufficient condition for \(Q\)’ or ‘\(Q\) is a necessary condition for \(P\)’.
7. Suppose for every \( x_0 \in (a, b) \), \( \lim_{x \to x_0} f(x) \) exists and is not \( \pm \infty \). What is the strongest conclusion you may draw?

(A) \( f \) is bounded on \((a, b)\).
(B) \( f \) is continuous on \((a, b)\).
(C) \( f \) attains its supremum and infimum on \((a, b)\).
(D) \( f \) satisfies the intermediate value property on \((a, b)\).
(E) None of the preceding.

**Answer:** Consider the case \((a, b) = (0, 1)\) and say

\[
f(x) = \begin{cases} 
  1/x & \text{if } x \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1), \\
  -2 & \text{if } x = \frac{1}{2}.
\end{cases}
\]

Note that \( f \) is continuous on \((0, \frac{1}{2}) \cup (\frac{1}{2}, 1)\) with a jump discontinuity at \( x = \frac{1}{2} \). So \( \lim_{x \to x_0} f(x) \) exists for any \( x_0 \in (0, 1) \). However \( f \) is unbounded and is discontinuous on \((0, 1)\). \( \sup_{x \in (0, 1)} = f(x) = +\infty \) and is not attained by any \( x_0 \in (0, 1) \). \( \inf_{x \in (0, 1)} f(x) = -2 < -1 < +\infty = \sup_{x \in (0, 1)} = f(x) \) but there is no \( x_0 \in (0, 1) \) such that \( f(x_0) = -1 \). Hence the answer is (E).

8. Which one of the following statements is correct?

(A) If \( \lim_{x \to x_0} f(x) \geq \lim_{x \to x_0} g(x) \), then there exists \( \delta > 0 \) such that \( f(x) \geq g(x) \) whenever \( 0 < |x - x_0| < \delta \).
(B) If \( \lim_{x \to x_0} f(x) > \lim_{x \to x_0} g(x) \), then there exists \( \delta > 0 \) such that \( f(x) > g(x) \) whenever \( 0 < |x - x_0| < \delta \).
(C) If there exists \( \delta > 0 \) such that \( f(x) > g(x) \) whenever \( 0 < |x - x_0| < \delta \), then \( \lim_{x \to x_0} f(x) \geq \lim_{x \to x_0} g(x) \).
(D) If there exists \( \delta > 0 \) such that \( f(x) > g(x) \) whenever \( 0 < |x - x_0| < \delta \) and that both \( \lim_{x \to x_0} f(x) \) and \( \lim_{x \to x_0} g(x) \) exist, then \( \lim_{x \to x_0} f(x) > \lim_{x \to x_0} g(x) \).
(E) None of the above.

**Answer:** (A) is incorrect in general; e.g. \( f(x) = -x \) and \( g(x) = x \), then \( \lim_{x \to 0} f(x) = 0 \geq 0 = \lim_{x \to 0} g(x) \) but there is no \( \delta > 0 \) such that \( f(x) \geq g(x) \) whenever \( 0 < |x - 0| < \delta \). (C) is incorrect in general since \( \lim_{x \to x_0} f(x) \) or \( \lim_{x \to x_0} g(x) \) may not exist; e.g. \( f(x) = 1/x^2 \) and \( g(x) = 1/x \) and \( x_0 = 0 \). (D) is incorrect in general; e.g. \( f(x) = x > x^2 = g(x) \) whenever \( 0 < |x - 0| < 1 \) but \( \lim_{x \to 0} f(x) = 0 \neq \lim_{x \to 0} g(x) \). (B) is the correct answer: Write \( \lim_{x \to x_0} f(x) = L \) and \( \lim_{x \to x_0} g(x) = M \) and pick \( \epsilon = (L-M)/2 > 0 \), then there exists \( \delta > 0 \) such that \( f(x) > L - \epsilon \) and \( g(x) < M + \epsilon \) whenever \( 0 < |x-x_0| < \delta \). Note that \( f(x) > L - \epsilon = (L+M)/2 = M+\epsilon > g(x) \).

9. Which one of the following statements regarding \( f(x) = 1/(1 + e^{1/x}) \) is correct?

(A) \( f \) has a removable discontinuity at 0.
(B) \( f \) has a jump discontinuity at 0.
(C) \( f \) has a discontinuity of the second kind at 0.
(D) \( f \) has unbounded at 0.
(E) \( f \) is continuous at 0.

**Answer:** Since

\[
\lim_{x \to 0^-} f(x) = \lim_{y \to -\infty} \frac{1}{1 + e^y} = 1 \quad \text{and} \quad \lim_{x \to 0^+} f(x) = \lim_{y \to +\infty} \frac{1}{1 + e^y} = 0,
\]

the correct answer is (B).
10. Suppose \( f \) is continuous on \((0, \infty)\), \( \lim_{x \to 0^+} f(x) = 0 \), and \( \lim_{x \to \infty} f(x) = 1 \). Which of the following statements is always true?

(A) There exists \( x_0 \in (0, \infty) \) such that \( f(x_0) = \frac{2}{\sqrt{3}} \).

(B) There exists \( x_0 \in (0, \infty) \) such that \( f(x_0) = -\frac{\sqrt{3}}{2} \).

(C) The infimum of \( f \) is 0 and the supremum of \( f \) is 1 on \((0, \infty)\).

(D) \( f \) is bounded on \((0, \infty)\).

(E) \( f \) is always nonnegative on \((0, \infty)\).

**Answer:** A counter example to (A) and (B) is given by
\[
 f(x) = \frac{x}{1 + x}
\]
since for all \( x > 0 \), \( 0 < f(x) < 1 \) and so never takes the values \( \frac{2}{\sqrt{3}} > 1 \) or \( -\frac{\sqrt{3}}{2} < 0 \). Define the function (draw the graph to see what it’s like):
\[
 t(x) = \begin{cases} 
 1 - x & \text{if } 1 \leq x \leq 2, \\
 x - 3 & \text{if } 2 \leq x \leq 3, \\
 0 & \text{otherwise.}
\end{cases}
\]

A counter example to (C) and (E) is
\[
 f(x) = \frac{x}{1 + x} + t(x)
\]
since \( f(2) = 1/2 - 1 = -1/2 \). The correct answer is (D): pick \( \varepsilon = 1 \), there exists \( K > 0 \) such that \( |f(x) - 1| < 1 \) (i.e. \( |f(x)| < 2 \)) whenever \( x \in [K, \infty) \) and there exists \( \delta > 0 \) such that \( |f(x) - 0| < 1 \) whenever \( x \in (0, \delta) \). Since \( f \) is continuous on \([\delta, K]\), there exists \( M > 0 \) such that \( |f(x)| < M \) whenever \( x \in [\delta, K] \) by Theorem 4.4. Hence \( |f(x)| < \max(M, 2) \) for all \( x \in [0, \infty) \).

11. Suppose there exists \( \delta > 0 \) such that \( f(x) \leq g(x) \) whenever \(-\delta < x - a < 0\). If \( \lim_{x \to a^-} f(x) \) and \( \lim_{x \to a^-} g(x) \) both exist, prove using \( \varepsilon-\delta \) definition that \( \lim_{x \to a^-} f(x) \leq \lim_{x \to a^-} g(x) \).

**Answer:** This proof is identical to the proof of Theorem 3.5 with \( N^*_\delta(a) \) replaced by \((a-\delta, a)\).