MATH 104: INTRODUCTORY ANALYSIS
SPRING 2010/11
MIDTERM EXAM II SOLUTIONS

1. Which one of the following functions is continuous at 0?

$$e(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}, \end{cases} \qquad f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}, \end{cases} \qquad g(x) = \begin{cases} |x| & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}, \end{cases}$$

$$h(x) = \begin{cases} q & \text{if } x = p/q,\ \gcd(p,q) = 1,\ q > 0, \\ 0 & \text{if } x \notin \mathbb{Q}, \end{cases}$$

(**A**) e and f only (**B**) e and h only (**C**) f and g only (**D**) g and h only
(**E**) none of them

ANSWER: By Examples **3.13**, **3.14**, and **4.4**, e is discontinuous at 0 while f is continuous at 0. Since the absolute value function $|\cdot|$ is continuous at 0, $g = f \circ |\cdot|$ is continuous at 0 by Theorem **4.3**. By Exercise **3.16**, h is unbounded at 0 and is therefore discontinuous at 0 by the Lemma on pp. 72. Hence the answer is (**C**).

2. f, g, h are all discontinuous at x_0. Which of the following could be continuous at x_0?

(**A**) $f \circ (g + h)$ (**B**) $f \cdot (g + h)$ (**C**) $f \circ (g \cdot h)$ (**D**) $f \circ (g \circ h)$ (**E**) all of them

ANSWER: Consider $x_0 = 0$. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \le 0, \end{cases}$$

and let $h = -g$. So $g + h = 0$ and $g \circ h = 0$. So for any f,

$$f \circ (g + h) = f \cdot (g + h) = f \circ (g \circ h) = 0$$

and are continuous. Note that $g \cdot h = h$. So if we pick $f = g$, then $f \circ (g \cdot h) = g \circ h = 0$ and is also continuous. So the answer is **(E)**.

3. f is continuous on $\mathbb{R}$ and $[f(x)]^4 = x^4$ for all $x \in \mathbb{Q}$. A possible value of $f(\sqrt{2})$ is:

(**A**) 4 (**B**) 2 (**C**) -2 (**D**) $-\sqrt{2}$ (**E**) any value in $\mathbb{R}$ is possible

ANSWER: By Exercise **4.13**, $[f(x)]^4 = x^4$ for all $x \in \mathbb{R}$. Note that $[f(x)]^4 = x^4$ implies that $[f(x)]^2 = x^2$. So by Exercise **4.17**, the four possibilities of f are $-x$, x, $-|x|$, or $|x|$. The only possible answer is (**D**).

4. What is the value of $\lim_{n\to\infty} n \sin(2/n)$?

(**A**) 0 (**B**) 1 (**C**) 2 (**D**) ∞ (**E**) limit does not exist

Answer: By Exercise **3.29**,

$$\lim_{x\to 0}\frac{\sin x}{x}=1.$$

By Theorem **3.6**, if we choose $x_n = 2/n$ and note that $x_n \to 0$, then

$$\lim_{n\to\infty}\frac{\sin x_n}{x_n}=1.$$

Now note that

$$\lim_{n\to\infty} n\sin\left(\frac{2}{n}\right) = 2\lim_{n\to\infty}\frac{\sin(2/n)}{2/n}=2.$$

Alternatively apply Exercise **3.30**(b). So the answer is **(C)**.

5. What is the value of the following?

$$\lim_{x\to 0}\frac{\tan x+\sqrt{4+x}-2}{x}$$

(**A**) 1 (**B**) 1/2 (**C**) 1/4 (**D**) 0 (**E**) limit does not exist

Answer: Note that by Theorem **3.4**(b) and Exercise **3.29**,

$$\lim_{x\to 0}\frac{\tan x}{x}=\lim_{x\to 0}\frac{\sin x}{x}\times\lim_{x\to 0}\frac{1}{\cos x}=1.$$

Note that as in Exercise **3.23**(a),

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{4+x}-2}{x} &= \lim_{x\to 0}\frac{(\sqrt{4+x}-2)(\sqrt{4+x}+2)}{x(\sqrt{4+x}+2)}\\ &= \lim_{x\to 0}\frac{4+x-4}{x(\sqrt{4+x}+2)}\\ &= \lim_{x\to 0}\frac{1}{\sqrt{4+x}+2}=\frac{1}{4}.\end{aligned}$$

Alternatively, use l'hôpital's rule. Hence by Theorem **3.4**(a),

$$\lim_{x\to 0}\frac{\tan x+\sqrt{4+x}-2}{x}=1+\frac{1}{4}=\frac{5}{4}.$$

None of the given answers are correct.

6. What can you say[1] about the following statement: "For any $\varepsilon\in(0,2)$, there exists $\delta>0$ such that $|f(x)-3|<4\varepsilon$ whenever $0<|x-5|<\delta$."

(**A**) This is a necessary but not sufficient condition for $\lim_{x\to 5} f(x)=3$.
(**B**) This is a sufficient but not necessary condition for $\lim_{x\to 5} f(x)=3$.
(**C**) This is both a necessary and a sufficient condition for $\lim_{x\to 5} f(x)=3$.
(**D**) This is neither a necessary nor a sufficient condition for $\lim_{x\to 5} f(x)=3$.
(**E**) This condition contradicts $\lim_{x\to 5} f(x)=3$.

Answer: Let P_1 be the statement "For any $\varepsilon\in(0,2)$, there exists $\delta>0$ such that $|f(x)-3|<4\varepsilon$ whenever $0<|x-5|<\delta$" and P_2 be the statement "For any $\varepsilon>0$, there exists $\delta>0$ such that $|f(x)-3|<\varepsilon$ whenever $0<|x-5|<\delta$". We claim that $P_1\Leftrightarrow P_2$, i.e. the answer is **(C)**. It is clear that $P_2\Rightarrow P_1$. Now assuming P_1. Given any $\varepsilon>0$, if $\varepsilon<8$, then let $\varepsilon'=\varepsilon/4\in(0,2)$ and so there exists $\delta>0$ such that $|f(x)-3|<4\varepsilon'=\varepsilon$ whenever $0<|x-5|<\delta$; if $\varepsilon\geq 8$, then let $\varepsilon'=1\in(0,2)$ (any other choice of $\varepsilon'\in(0,2)$ would work too) and so again so there exists

[1] Recall if $P\Rightarrow Q$, we say 'P is a *sufficient* condition for Q' or 'Q is a *necessary* condition for P'.

$\delta > 0$ such that $|f(x) - 3| < 4\varepsilon' = 4 < 8 \le \varepsilon$ whenever $0 < |x - 5| < \delta$. Hence for whatever given value of $\varepsilon > 0$, P_2 must hold if P_1 holds.

7. Suppose for every $x_0 \in (a, b)$, $\lim_{x \to x_0} f(x)$ exists and is not $\pm\infty$. What is the strongest conclusion you may draw?

(**A**) f is bounded on (a, b). (**B**) f is continuous on (a, b).
(**C**) f attains its supremum and infimum on (a, b).
(**D**) f satisfies the intermediate value property on (a, b). (**E**) None of the preceding.

Answer: Consider the case $(a, b) = (0, 1)$ and say

$$f(x) = \begin{cases} 1/x & \text{if } x \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1), \\ -2 & \text{if } x = \frac{1}{2}. \end{cases}$$

Note that f is continuous on $(0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ with a jump discontinuity at $x = \frac{1}{2}$. So $\lim_{x \to x_0} f(x)$ exists for any $x_0 \in (0, 1)$. However f is unbounded and is discontinuous on $(0, 1)$. $\sup_{x \in (0,1)} = f(x) = +\infty$ and is not attained by any $x_0 \in (0, 1)$. $\inf_{x \in (0,1)} f(x) = -2 < -1 < +\infty = \sup_{x \in (0,1)} = f(x)$ but then there is no $x_0 \in (0, 1)$ such that $f(x_0) = -1$. Hence the answer is **(E)**.

8. Which one of the following statements is correct?

(**A**) If $\lim_{x \to x_0} f(x) \ge \lim_{x \to x_0} g(x)$, then there exists $\delta > 0$ such that $f(x) \ge g(x)$ whenever $0 < |x - x_0| < \delta$.
(**B**) If $\lim_{x \to x_0} f(x) > \lim_{x \to x_0} g(x)$, then there exists $\delta > 0$ such that $f(x) > g(x)$ whenever $0 < |x - x_0| < \delta$.
(**C**) If there exists $\delta > 0$ such that $f(x) > g(x)$ whenever $0 < |x - x_0| < \delta$, then $\lim_{x \to x_0} f(x) \ge \lim_{x \to x_0} g(x)$.
(**D**) If there exists $\delta > 0$ such that $f(x) > g(x)$ whenever $0 < |x - x_0| < \delta$ and that both $\lim_{x \to x_0} f(x)$ and $\lim_{x \to x_0} g(x)$ exist, then $\lim_{x \to x_0} f(x) > \lim_{x \to x_0} g(x)$.
(**E**) None of the above.

Answer: **(A)** is incorrect in general; e.g. $f(x) = -x$ and $g(x) = x$, then $\lim_{x \to 0} f(x) = 0 \ge 0 = \lim_{x \to 0} g(x)$ but there is no $\delta > 0$ such that $f(x) \ge g(x)$ whenever $0 < |x - 0| < \delta$. **(C)** is incorrect in general since $\lim_{x \to x_0} f(x)$ or $\lim_{x \to x_0} g(x)$ may not exist; e.g. $f(x) = 1/x^2$ and $g(x) = 1/x$ and $x_0 = 0$. **(D)** is incorrect in general; e.g. $f(x) = x > x^2 = g(x)$ whenever $0 < |x - 0| < 1$ but $\lim_{x \to 0} f(x) = 0 = \lim_{x \to 0} g(x)$. **(B)** is the correct answer: Write $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} g(x) = M$ and pick $\varepsilon = (L - M)/2 > 0$, then there exists $\delta > 0$ such that $f(x) > L - \varepsilon$ and $g(x) < M + \varepsilon$ whenever $0 < |x - x_0| < \delta$. Note that $f(x) > L - \varepsilon = (L + M)/2 = M + \varepsilon > g(x)$.

9. Which one of the following statements regarding $f(x) = 1/(1 + e^{1/x})$ is correct?

(**A**) f has a removable discontinuity at 0. (**B**) f has a jump discontinuity at 0.
(**C**) f has a discontinuity of the second kind at 0. (**D**) f has unbounded at 0.
(**E**) f is continuous at 0.

Answer: Since

$$\lim_{x \to 0^-} f(x) = \lim_{y \to -\infty} \frac{1}{1 + e^y} = 1 \quad \text{and} \quad \lim_{x \to 0^+} f(x) = \lim_{y \to +\infty} \frac{1}{1 + e^y} = 0,$$

the correct answer is **(B)**.

10. Suppose f is continuous on $(0,\infty)$, $\lim_{x\to 0^+} f(x) = 0$, and $\lim_{x\to\infty} f(x) = 1$. Which of the following statements is always true?

(**A**) There exists $x_0 \in (0,\infty)$ such that $f(x_0) = 2/\sqrt{3}$.
(**B**) There exists $x_0 \in (0,\infty)$ such that $f(x_0) = -\sqrt{3}/2$.
(**C**) The infimum of f is 0 and the supremum of f is 1 on $(0,\infty)$.
(**D**) f is bounded on $(0,\infty)$. (**E**) f is always nonnegative on $(0,\infty)$.

Answer: A counter example to **(A)** and **(B)** is given by

$$f(x) = \frac{x}{1+x}$$

since for all $x > 0$, $0 < f(x) < 1$ and so never takes the values $2/\sqrt{3} > 1$ or $-\sqrt{3}/2 < 0$. Define the function (draw the graph to see what it's like):

$$t(x) = \begin{cases} 1-x & \text{if } 1 \le x \le 2, \\ x-3 & \text{if } 2 \le x \le 3, \\ 0 & \text{otherwise.} \end{cases}$$

A counter example to **(C)** and **(E)** is

$$f(x) = \frac{x}{1+x} + t(x)$$

since $f(2) = 1/2 - 1 = -1/2$. The correct answer is **(D)**: pick $\varepsilon = 1$, there exists $K > 0$ such that $|f(x) - 1| < 1$ (i.e. $|f(x)| < 2$) whenever $x \in [K,\infty)$ and there exists $\delta > 0$ such that $|f(x) - 0| < 1$ whenever $x \in (0,\delta)$. Since f is continuous on $[\delta, K]$, there exists $M > 0$ such that $|f(x)| < M$ whenever $x \in [\delta, K]$ by Theorem **4.4**. Hence $|f(x)| < \max(M, 2)$ for all $x \in [0,\infty)$.

11. Suppose there exists $\delta > 0$ such that $f(x) \le g(x)$ whenever $-\delta < x - a < 0$. If $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^-} g(x)$ both exist, prove using ε-δ definition that $\lim_{x\to a^-} f(x) \le \lim_{x\to a^-} g(x)$.

Answer: This proof is identical to the proof of Theorem **3.5** with $N^*_\delta(a)$ replaced by $(a-\delta, a)$.