MATH 104: INTRODUCTORY ANALYSIS
SPRING 2010/11
MIDTERM EXAM I

Instructions. There are **two** printed pages. For questions **1–10**, circle the correct answer. For question **11**, attach additional paper if necessary. This exam is closed book, no cheat sheet.

1. Which of the following sequences are Cauchy?

$$a_n = 1 + r + r^2 + \cdots + r^n, \quad b_n = \frac{4n^3 + 3n^2 + 2n + 1}{7n^3 - 2n^2 + 3n}, \quad c_n = \left(1 - \frac{1}{n}\right) \sin \frac{n\pi}{2}$$

where $r < -2/\sqrt{3}$ and $n \in \mathbb{N}$.

(**A**) $(a_n)_{n\in\mathbb{N}}$ only (**B**) $(b_n)_{n\in\mathbb{N}}$ only (**C**) $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ only
(**D**) $(a_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ only (**E**) none of them

ANSWER: $(a_n)_{n\in\mathbb{N}}$ is the sequence on pp. **37**, it diverges since $|r| > 1$ and therefore is not Cauchy. $(b_n)_{n\in\mathbb{N}}$ is the sequence in Example **2.4**, which converges and is therefore Cauchy. $(c_n)_{n\in\mathbb{N}}$ is the sequence in Example **2.8**, which is not convergent and therefore not Cauchy. So the answer is **(B)**.

2. Which of the following sequences are bounded?

$$a_n = 1 + r + r^2 + \cdots + r^n, \quad b_n = \left(2 + \frac{1}{n}\right)^n, \quad c_n = \left(\frac{\sin n + \cos n}{3}\right)^n$$

where $-1/\sqrt{2} < r < \sqrt{3}/2$ and $n \in \mathbb{N}$.

(**A**) $(a_n)_{n\in\mathbb{N}}$ only (**B**) $(b_n)_{n\in\mathbb{N}}$ only (**C**) $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ only
(**D**) $(a_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ only (**E**) none of them

ANSWER: $(a_n)_{n\in\mathbb{N}}$ is the sequence on pp. **37**, it converges since $|r| < 1$ and is therefore bounded. $(b_n)_{n\in\mathbb{N}}$ is the sequence in Exercise **2.1(e)**, which is unbounded since

$$b_n = \left(2 + \frac{1}{n}\right)^n \geq 2^n.$$

$(c_n)_{n\in\mathbb{N}}$ is the sequence in Exercise **2.1(g)**, which is converges to 0 by Squeezing Lemma since

$$0 \leq |c_n| = \left|\frac{\sin n + \cos n}{3}\right|^n \leq \left(\frac{|\sin n| + |\cos n|}{3}\right)^n \leq \left(\frac{2}{3}\right)^n,$$

and is therefore bounded. So the answer is **(D)**.

3. What is the value of the following?

$$\lim_{n\to\infty} \frac{1}{n}\left[\left(2 - \frac{1}{2}\right) + \left(2 - \frac{1}{2^2}\right) + \cdots + \left(2 - \frac{1}{2^n}\right)\right]$$

(**A**) 0 (**B**) 1 (**C**) 2 (**D**) e^2 (**E**) the limit does not exist

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Answer: This follows from the result in Exercise **2.10** in Homework **2**,

$$\lim_{n\to\infty}\frac{1}{n}\left[\left(2-\frac{1}{2}\right)+\left(2-\frac{1}{2^2}\right)+\cdots+\left(2-\frac{1}{2^n}\right)\right]=\lim_{n\to\infty}\left(2-\frac{1}{2^n}\right)=2$$

where the second sequence is the one in Example **2.2**. So the answer is **(C)**.

4. What is the value of the following?

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{n+\frac{1}{n}}$$

(**A**) 0 (**B**) 1 (**C**) e (**D**) $1/e$ (**E**) $e+1/e$

Answer: By Theorem **2.3(c)**,

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{n+\frac{1}{n}}=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{n}\times\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{\frac{1}{n}}.$$

The first sequence is the one in Exercise **2.13(g)** and can be evaluated via Theorem **2.3(e)**,

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{n}=\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^{n}=\frac{\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^{n+1}}{\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)}=e^{-1}.$$

The second sequence is the one in Exercise **2.1(f)** and can be evaluated to be 1 via Squeezing Lemma

$$\left(\frac{1}{2}\right)^{1/n}\le\left(\frac{n}{n+1}\right)^{1/n}\le 1$$

and Example **2.5**. So the answer is **(D)**.

5. For each $n\in\mathbb{N}$, let

$$c_n=\left(1-\frac{1}{n}\right)\sin\frac{n\pi}{2}.$$

How many convergent subsequences does the sequence $(c_n)_{n\in\mathbb{N}}$ have?

(**A**) none (**B**) one (**C**) two (**D**) three (**E**) infinitely many

Answer: Any sequence with at least one convergent subsequence will have infinitely many convergent subsequences. So the answer is **(E)**.

6. Suppose we have a sequence of non-empty closed intervals $I_1=[a_1,b_1], I_2=[a_2,b_2], I_3=[a_3,b_3],\ldots$, and suppose that they are nested, i.e. $I_1\supseteq I_2\supseteq I_3\supseteq\cdots$. Which of the following statement is always true?

(**A**) There is at least one real number common to all the intervals I_n
(**B**) There is at most one real number common to all the intervals I_n
(**C**) There is exactly one real number common to all the intervals I_n
(**D**) There may not be any real number common to all the intervals I_n
(**E**) All of the above could happen — it depends on the choice of I_n

Answer: This looks like Theorem **2.6** except that it is lacking the condition $\lim_{n\to\infty}(b_n-a_n)=0$. Just as in the proof of Theorem **2.6**, $(a_n)_{n\in\mathbb{N}}$ is a monotone increasing sequence bounded above by b_1 and so is convergent. The limit $A=\lim_{n\to\infty}a_n\le b_k$ for all $k\in\mathbb{N}$ and so $A\in I_k$

for all $k \in \mathbb{N}$. We may eliminate **(B)** and **(C)** by considering the intervals $I_n = [-1 - \frac{1}{n}, 1 + \frac{1}{n}]$, any $-1 \le a < 1$ is common to all these I_n. So the answer is **(A)**.

7. Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be sequences with where $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} b_n = 0$. Which of the following statement is always true?

(**A**) $\lim_{n\to\infty} a_n b_n = 0$ (**B**) $\lim_{n\to\infty} a_n b_n = 1$ (**C**) $\lim_{n\to\infty} a_n b_n = \infty$
(**D**) $\lim_{n\to\infty} a_n b_n$ does not exist and is not ∞
(**E**) All of the above could happen — it depends on the choice of a_n and b_n

ANSWER: This is just Exercise **2.7**. The answer is **(E)**.

8. Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be sequences with where $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} b_n = 0$. Which of the following statement is always true?

(**A**) $\lim_{n\to\infty} a_n + b_n = 0$ (**B**) $\lim_{n\to\infty} a_n + b_n = 1$ (**C**) $\lim_{n\to\infty} a_n + b_n = \infty$
(**D**) $\lim_{n\to\infty} a_n + b_n$ does not exist and is not ∞
(**E**) All of the above could happen — it depends on the choice of a_n and b_n

ANSWER: This is a variant of Exercise **2.4**. The answer is **(C)**.

9. Let $(a_n)_{n\in\mathbb{N}}$ be a monotone increasing sequence where $a_n < M$ for all $n \in \mathbb{N}$. Which of the following statement is always true?

(**A**) $\sup_{n\in\mathbb{N}} a_n < M$ (**B**) $\inf_{n\in\mathbb{N}} a_n < M$ (**C**) $\limsup_{n\to\infty} a_n < M$
(**D**) $\liminf_{n\to\infty} a_n < M$ (**E**) $\lim_{n\to\infty} a_n < M$

ANSWER: The answer is **(B)** since for a monotone increasing sequence,

$$\inf_{n\in\mathbb{N}} a_n = a_1 < M.$$

By Theorem **2.5**, $\lim_{n\to\infty} a_n$ exists and equals $\sup_{n\in\mathbb{N}} a_n$. By the corollary to Theorem **2.9**,

$$\limsup_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = \lim_{n\to\infty} a_n = \sup_{n\in\mathbb{N}} a_n.$$

So **(A)**, **(C)**, **(D)**, **(E)** are really identical. Let $M = 1$. The sequence

$$a_n = 1 - \frac{1}{n}$$

satisfies the condition of the problem but **(A)**, **(C)**, **(D)**, **(E)** are all false (the limit is not *strictly* less than M).

10. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence where

$$\liminf_{n\to\infty} a_n = -1 \quad \text{and} \quad \limsup_{n\to\infty} a_n = 1.$$

Which of the following statement must be true?

(**A**) $(a_n)_{n\in\mathbb{N}}$ is monotone increasing (**B**) $(a_n)_{n\in\mathbb{N}}$ is strictly increasing
(**C**) $(a_n)_{n\in\mathbb{N}}$ is bounded (**D**) $(a_n)_{n\in\mathbb{N}}$ is Cauchy (**E**) None of them

ANSWER: By Theorem **2.9** and following the proof of the Lemma on pp. **40**, the answer is **(C)**.

11. Let $(a_n)_{n\in\mathbb{N}}$ be convergent and $b \in \mathbb{R}$. Suppose $a_n \geq b$ for all $n \in \mathbb{N}$. Show that $\lim_{n\to\infty} a_n \geq b$.

ANSWER: Suppose not and $\lim_{n\to\infty} a_n = a < b$. Let $\varepsilon = b - a > 0$. Then there exists $N \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ whenever $n > N$. So if $n > N$, then $a - \varepsilon < a_n < a + \varepsilon$ and in particular

$$a_n < a + \varepsilon = a + (b - a) = b,$$

contradicting $a_n \geq b$.