MATH 104: INTRODUCTORY ANALYSIS
SPRING 2009/10
PROBLEM SET 8 SOLUTIONS

6.9: Let

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \cap [a, b], \\ 0 & \text{if } x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [a, b], \end{cases} \qquad \text{where } a > 0.$$

Prove that

$$\overline{\int} f = \frac{b^3 - a^3}{3}.$$

Solution. Let $P_n = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a regular partition of $[a, b]$ into n subintervals, i.e.

$$\Delta x_i = \frac{b-a}{n} \quad \text{and} \quad x_i = a + i\left(\frac{b-a}{n}\right)$$

for $i = 1, \ldots, n$. Hence

$$\begin{aligned} U(P_n, f) &= \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} x_i^2 \left(\frac{b-a}{n}\right) \\ &= \frac{b-a}{n} \sum_{i=1}^{n} \left[a + i\left(\frac{b-a}{n}\right)\right]^2 \\ &= \frac{b-a}{n} \sum_{i=1}^{n} \left[a^2 + 2ai\left(\frac{b-a}{n}\right) + i^2\left(\frac{b-a}{n}\right)^2\right] \\ &= \frac{b-a}{n} \sum_{i=1}^{n} a^2 + 2a\left(\frac{b-a}{n}\right)^2 \sum_{i=1}^{n} i + \left(\frac{b-a}{n}\right)^3 \sum_{i=1}^{n} i^2 \\ &= (b-a)a^2 + 2a\left(\frac{b-a}{n}\right)^2 \frac{n(n+1)}{2} + \left(\frac{b-a}{n}\right)^3 \frac{n(n+1)(2n+1)}{6} - \frac{b^3 - a^3}{3} \\ &= \frac{b^3 - a^3}{3} + \frac{(b^2 - a^2)(b-a)}{2n} + \frac{(b-a)^3}{6n^2} \end{aligned}$$

Note that we have used Exercises **1.13** and **1.14** for the sums. Since $\overline{\int} f \le U(P, f)$ for every partition P, it follows that

$$\overline{\int_a^b} f \le \frac{b^3 - a^3}{3} + \frac{(b^2 - a^2)(b-a)}{2n} + \frac{(b-a)^3}{6n^2}$$

for all $n \in \mathbb{N}$. Therefore

$$\overline{\int_a^b} f \le \frac{b^3 - a^3}{3}. \tag{1}$$

By Theorem **6.4**, given any $\varepsilon > 0$, there exists $\delta > 0$ such that $U(P, f) \le \overline{\int} f + \varepsilon$ whenever $\|P\| < \delta$. Since for n large enough, we will have $\|P_n\| = (b-a)/n < \delta$, so

$$U(P_n, f) \le \overline{\int_a^b} f + \varepsilon$$

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and so

$$\frac{b^3-a^3}{3}+\frac{(b^2-a^2)(b-a)}{2n}+\frac{(b-a)^3}{6n^2}\le\overline{\int_a^b} f+\varepsilon.$$

Therefore

$$\frac{b^3-a^3}{3}\le\overline{\int_a^b} f+\varepsilon$$

and by the arbitrariness of ε, it follows that

$$\frac{b^3-a^3}{3}\le\overline{\int_a^b} f. \tag{2}$$

Hence by (1) and (2),

$$\overline{\int_a^b} f=\frac{b^3-a^3}{3}.$$

6.12: Prove that the following function is Riemann-integrable on $[0,1]$ even though it has infinitely many discontinuities on $[0,1]$:

$$f(x)=\begin{cases}1 & \text{if } x=\dfrac{1}{n} \text{ where } n=1,2,3,\ldots,\\ 0 & \text{otherwise.}\end{cases}$$

Solution. Let $\varepsilon>0$. There exists $N\in\mathbb{N}$ such that $1/n<\varepsilon/2$ whenever $n>N$. Let P be a partition of $[0,1]$ where

$$0=x_0<x_1=\frac{1}{N+1}<x_2<\cdots<x_k=1$$

and such that

$$\Delta x_i=x_i-x_{i-1}<\frac{\varepsilon}{4N}\quad\text{for } i=2,3,\ldots,k.$$

So

$$U(P,f)=\sum_{i=1}^{k}M_i\Delta x_i\le\frac{1}{N+1}+N\times\frac{\varepsilon}{4N}<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

since at most $2N$ of the numbers $M_2,\ldots,M_k$ could be 1 and the rest have to be 0 (because $[x_1,1]=[1/(N+1),1]$ contains only $1/N,1/(N-1),\ldots,1/2,1$ where f takes the value 1 — the worst that could happen is if all these occur among $x_2,\ldots,x_k$). Note that

$$L(P,f)=\sum_{i=1}^{n}m_i\Delta x_i=0$$

since $m_i=0$ for all $i=1,2,\ldots,k$. Hence

$$U(P,f)-L(P,f)<\varepsilon$$

and so f is Riemann-integrable by Theorem **6.1**.

6.15: Using Definition **6.3** and Theorem **6.8**, prove that if c is any real number then

$$\int_a^b f=\int_a^c f+\int_c^b f$$

provided f is Riemann-integrable on the largest of the intervals $[a,b]$, $[c,b]$, and $[a,c]$.

Solution. If $a<c<b$, the proof of this is given in Theorem **6.8**. For the case $a<b\le c$, Theorem **6.8** again yields

$$\int_a^c f=\int_a^b f+\int_b^c f$$

and so

$$\int_a^b f = \int_a^c f - \int_b^c f = \int_a^c f + \int_c^b f$$

by Definition **6.3**. Likewise for the case $c \le a < b$.

6.20: Prove Theorem **6.13**. *Hint*: First prove that the result holds when f and g are nonnegative integable functions on $[a,b]$ and then use the identity

$$fg = (f^+ - f^-)(g^+ - g^-) = f^+g^+ - f^+g^- - f^-g^+ + f^-g^-$$

to establish Theorem **6.13**.

Solution. Suppose f and g are nonnegative and Riemann integrable on $[a,b]$. Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition. For any bounded function $F : [a,b] \to \mathbb{R}$, denote

$$m_i(F) = \inf_{x\in[x_{i-1},x_i]} F(x) \quad \text{and} \quad M_i(F) = \sup_{x\in[x_{i-1},x_i]} F(x),$$
$$m(F) = \inf_{x\in[a,b]} F(x) \quad \text{and} \quad M(F) = \sup_{x\in[a,b]} F(x).$$

For nonnegative F,

$$0 \le m(F) \le m_i(F) \le M_i(F) \le M(F).$$

Now since f and g are nonnegative,

$$M_i(fg) \le M_i(f)M_i(g) \quad \text{and} \quad m_i(f)m_i(g) \le m_i(fg).$$

So

$$\begin{aligned}
U(P,fg) - L(P,fg) &= \sum_{i=1}^n [M_i(fg) - m_i(fg)]\Delta x_i \\
&\le \sum_{i=1}^n [M_i(f)M_i(g) - m_i(f)m_i(g)]\Delta x_i \\
&= \sum_{i=1}^n [M_i(f)M_i(g) - M_i(f)m_i(g) + M_i(f)m_i(g) - m_i(f)m_i(g)]\Delta x_i \\
&= \sum_{i=1}^n M_i(f)[M_i(g) - m_i(g)]\Delta x_i + \sum_{i=1}^n m_i(g)[M_i(f) - m_i(f)]\Delta x_i \\
&\le M(f)\sum_{i=1}^n [M_i(g) - m_i(g)]\Delta x_i + M(g)\sum_{i=1}^n [M_i(f) - m_i(f)]\Delta x_i \\
&= M(f)[U(P,g) - L(P,g)] + M(g)[U(P,f) - L(P,f)]. \qquad (3)
\end{aligned}$$

Note that if $M(f)$ or $M(g) = 0$, then f or g is the zero function and therefore fg is the zero function and therefore Riemann integrable. So we will assume that both $M(f) > 0$ and $M(g) > 0$. Let $\varepsilon > 0$ be given. Since f and g are Riemann integrable, there exists P_1 such that

$$U(P_1,g) - L(P_1,g) < \frac{\varepsilon}{2M(f)}$$

and there exists P_2 such that

$$U(P_2,f) - L(P_2,f) < \frac{\varepsilon}{2M(g)}.$$

Let P be a common refinement of P_1 and P_2. Then by Lemma **2** on pp. 153,

$$U(P,g) - L(P,g) < U(P_1,g) - L(P_1,g) < \frac{\varepsilon}{2M(f)}$$

and

$$U(P,f) - L(P,f) < U(P_2,f) - L(P_2,f) < \frac{\varepsilon}{2M(g)}.$$

Hence by (3),

$$\begin{aligned} U(P,fg) - L(P,fg) &\leq M(f)[U(P,g) - L(P,g)] + M(g)[U(P,f) - L(P,f)] \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

and so fg is Riemann-integrable by Theorem **6.1**. Now for general f and g, we apply what we have just proved to deduce that $f^+g^+, f^+g^-, f^-g^+, f^-g^-$ (note that they are all products of two nonnegative functions) are Riemann-integrable. The we apply Theorem **6.6** to deduce that

$$f^+g^+ - f^+g^- - f^-g^+ + f^-g^-$$

is also Riemann-integrable. But by the hint, this is just fg.

6.22: Prove that if f and g are continuous on $[a,b]$ and if g does not change sign in $[a,b]$ then there is a point $c \in (a,b)$ such that

$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx \tag{4}$$

SOLUTION. We will assume WLOG that $g(x) \geq 0$ for all $x \in [a,b]$ (if not, consider $-g$). Since f is continuous on $[a,b]$, so f is bounded on $[a,b]$. Let

$$m = \inf_{x\in[a,b]} f(x) \quad \text{and} \quad M = \sup_{x\in[a,b]} f(x)$$

and so

$$m \leq f(x) \leq M.$$

Since $g(x) \geq 0$,

$$mg(x) \leq f(x)g(x) \leq Mg(x).$$

Hence by Theorem **6.9**,

$$m\int_a^b g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq M\int_a^b g(x)\,dx.$$

If $\int_a^b g(x)\,dx = 0$, then $\int_a^b f(x)g(x)\,dx = 0$ and so (4) is trivially satisfied by any c. Otherwise, set

$$k := \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx}$$

and note that $k \in [m,M]$. By Theorem **4.6** (IVT), there exists $c \in (a,b)$ such that $f(c) = k$.

6.23: If $f \in \mathcal{R}[a,b]$ define $F(x) = \int_a^x f$ for every $x \in [a,b]$. Prove that F is continuous on $[a,b]$.

SOLUTION. By Theorem **6.7**, since $f \in \mathcal{R}[a,b]$, therefore $f \in \mathcal{R}[a,c]$ for any $c \in [a,b]$. Hence $F(c)$ exists for all $c \in [a,b]$. We claim that for all $c \in (a,b)$,

$$\lim_{x\to c^+} F(x) = F(c) = \lim_{x\to c^-} F(x)$$

and also

$$\lim_{x\to a^-} F(x) = F(a), \quad \lim_{x\to b^+} F(x) = F(b),$$

and therefore F is continuous on $[a,b]$ by Theorem **3.7**. Since $f \in \mathcal{B}[a,b]$, there exists $M > 0$ such that $|f(x)| \le M$ for all $x \in [a,b]$. Let $\varepsilon > 0$ and set $\delta = \varepsilon/M$. If $c \le x < c+\delta$, then by Problem **6.15** and Theorem **6.12**,

$$\begin{aligned} |F(x) - F(c)| &= \left| \int_a^x f(t)\,dt - \int_a^c f(t)\,dt \right| \\ &= \left| \int_a^x f(t)\,dt + \int_c^a f(t)\,dt \right| \\ &= \left| \int_c^x f(t)\,dt \right| \\ &\le \int_c^x |f(t)|\,dt \\ &\le M(x-c) < M\delta = \varepsilon. \end{aligned}$$

This shows that $\lim_{x\to c^+} F(x) = F(c)$ for all $c \in [a,b)$. A essentially identical argument shows that $\lim_{x\to c^-} F(x) = F(c)$ for all $c \in (a,b]$.

6.25: Prove:

(1) If $f \in \mathcal{R}[0,b]$ and f is an even function then $f \in \mathcal{R}[-b,b]$ and

$$\int_{-b}^{b} f = 2\int_0^b f.$$

(2) If $f \in \mathcal{R}[0,b]$ and f is an odd function then $f \in \mathcal{R}[-b,b]$ and

$$\int_{-b}^{b} f = 0.$$

Solution. (*sketch*) We define a *symmetric partition* Q of $[-b,b]$ as a partition with the property that (i) $0 \in Q$; and the property that (ii) $x \in Q$ if and only $-x \in Q$. For any bounded function f on $[-b,b]$ (not necessarily even nor odd), we can show that

$$\underline{\int_{-b}^{b}} f = \sup\{L(Q,f) \mid Q \text{ symmetric partition of } [-b,b]\},$$

$$\overline{\int_{-b}^{b}} f = \inf\{U(Q,f) \mid Q \text{ symmetric partition of } [-b,b]\},$$

since given any nonsymmetric partition P of $[-b,b]$, we can refine it into a symmetric partition by simply taking $Q = P \cup (-P)$. Now it remains to observe that for an even function f and a symmetric partition Q of $[-b,b]$,

$$L(Q,f) = 2L(P,f), \quad U(Q,f) = 2U(P,f)$$

where $P = Q \cap [0,b]$ is a partition of $[0,b]$. Whereas for an odd function f and a symmetric partition Q of $[-b,b]$,

$$L(Q,f) = 0, \quad U(Q,f) = 0.$$

Riemann integrability and the required equations then follows.

6.27: Prove the *integration-by-parts formula*: if f,g are differentiable on $[a,b]$ and if f',g' are integrable on $[a,b]$ then

$$\int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b f'g.$$

Solution. By the product rule,

$$(fg)' = f'g + fg'. \tag{5}$$

Since f, g are differentiable and therefore continous and therefore Riemann integrable on $[a, b]$ and since f', g' are assumed to be Riemann integrable on $[a, b]$, it follows from Theorem **6.6** and Problem **6.20** that $(fg)'$ is Riemann integrable on $[a, b]$. Hence by Theorem **6.17**,

$$\int_a^b (fg)' = f(b)g(b) - f(a)g(a). \tag{6}$$

Now subsitute (5) into (6) and apply Theorem **6.6** to get the required formula.