MATH 104: INTRODUCTORY ANALYSIS
SPRING 2009/10
PROBLEM SET 7 SOLUTIONS

5.31: Prove that if f' is continuous in a neighborhood of $x = a$ then

$$\lim_{h\to 0} \frac{f(a+h/2) - f(a-h/2)}{h} = f'(a).$$

SOLUTION. Let $\delta > 0$ be such that f is differentiable and f' is continuous in $N_\delta(a)$. Define

$$F(h) = f(a+h/2) - f(a-h/2) \quad \text{and} \quad G(h) = h.$$

Then

(i) $F'(h) = \frac{1}{2}[f'(a+h/2) + f'(a-h/2)]$ and $G'(h) = 1$ exist in $N^*_\delta(a)$;
(ii) $G'(h) = 1 \neq 0$ in $N^*_\delta(a)$;
(iii) $\lim_{h\to a} F(h) = \lim_{h\to a} G(h) = 0$;
(iv) $\lim_{h\to a}[F'(h)/G'(h)]$ exists since

$$\lim_{h\to a} \frac{F'(h)}{G'(h)} = \lim_{h\to a} \frac{f'(a+h/2) + f'(a-h/2)}{2} = \frac{f'(a) + f'(a)}{2} = f'(a)$$

by the continuity of f' at a.

By l'hôpital rule, $\lim_{h\to a}[F(h)/G(h)]$ exists and

$$\lim_{h\to a} \frac{F(h)}{G(h)} = \lim_{h\to a} \frac{F'(h)}{G'(h)}.$$

Hence

$$\lim_{h\to 0} \frac{f(a+h/2) - f(a-h/2)}{h} = f'(a).$$

5.32: Prove that if f'' exists and is continuous in a neighborhood of $x = a$ then

$$\lim_{h\to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} = f''(a).$$

SOLUTION. Let $\delta_1 > 0$ be such that f is twice differentiable and f'' is continuous in $N_{\delta_1}(a)$. Let $\delta = \min\{\delta_1, |a|/2\}$ if $a \neq 0$ and $\delta = \delta_1$ if $a = 0$ (so that condition (ii) below is satisfied). Define

$$F(h) = f(a+h) - 2f(a) + f(a-h) \quad \text{and} \quad G(h) = h^2.$$

Then

(i) $F'(h) = f'(a+h) - f'(a-h)$ and $G'(h) = 2h$ exist in $N^*_\delta(a)$;
(ii) $G'(h) = 2h \neq 0$ in $N^*_\delta(a)$;
(iii) $\lim_{h\to a} F(h) = \lim_{h\to a} G(h) = 0$;
(iv) $\lim_{h\to a}[F'(h)/G'(h)]$ exists since

$$\lim_{h\to a} \frac{F'(h)}{G'(h)} = \lim_{h\to a} \frac{f'(a+h) - f'(a-h)}{2h} \overset{t=2h}{=} \lim_{t\to 0} \frac{f'(a+t/2) - f'(a-t/2)}{t} = f''(a)$$

where the last equality follows from part (a).

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By l'hôpital rule, $\lim_{h\to a}[F(h)/G(h)]$ exists and

$$\lim_{h\to a}\frac{F(h)}{G(h)} = \lim_{h\to a}\frac{F'(h)}{G'(h)}.$$

Hence

$$\lim_{h\to 0}\frac{f(a+h)-2f(a)+f(a-h)}{h^2} = f''(a).$$

5.33: Prove Theorem **5.12** using the mean-value theorem. If f is continuous at $x=a$ and if f is differentiable in $N^*_\delta(a)$ for some $\delta>0$ and if $\lim_{x\to a} f'(x)$ exists then $f'(a)$ exists and $f'(a)=\lim_{x\to a} f'(x)$.

SOLUTION. Note that if $0<h<\delta$, then f is continuous on $[a,a+h]$ and differentiable on $(a,a+h)$ so mean value theorem implies that there exists $c_h\in(a,a+h)$ such that

$$f'(c_h) = \frac{f(a+h)-f(a)}{h}.$$

Now note that $a<c_h<a+h$ implies that $\lim_{h\to 0^+} c_h = 0$. Hence

$$\lim_{h\to 0^+} f'(c_h) = \lim_{h\to 0^+}\frac{f(a+h)-f(a)}{h}.$$

Applying the same argument to f on the interval $[a-h,a]$, we get

$$\lim_{h\to 0^-} f'(c_h) = \lim_{h\to 0^-}\frac{f(a+h)-f(a)}{h}.$$

But since $\lim_{x\to a} f'(x)$ exists, we have

$$\lim_{h\to 0^-} f'(c_h) = \lim_{x\to a} f'(x) = \lim_{h\to 0^+} f'(c_h).$$

Hence

$$\lim_{h\to 0^-}\frac{f(a+h)-f(a)}{h} = \lim_{x\to a} f'(x) = \lim_{h\to 0^+}\frac{f(a+h)-f(a)}{h}$$

and so

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

exists and equals $\lim_{x\to a} f'(x)$. In other words, $f'(a)$ exists and $f'(a)=\lim_{x\to a} f'(x)$.

4.46: Prove that if f and g are each uniformly continuous on I then the sum $f+g$ is uniformly continuous on I.

SOLUTION. Let $\varepsilon>0$. There exists $\delta_1>0$ such that $|f(x)-f(y)|<\varepsilon/2$ whenever $|x-y|<\delta_1$. There exists $\delta_2>0$ such that $|g(x)-g(y)|<\varepsilon/2$ whenever $|x-y|<\delta_2$. Let $\delta=\min(\delta_1,\delta_2)$. Then whenever $|x-y|<\delta$, we have

$$|[f(x)+g(x)]-[f(y)+g(y)]| \le |f(x)-f(y)|+|g(x)-g(y)| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

4.48: Prove that if f is uniformly continuous on I and $k\in\mathbb{R}$ then $k\cdot f$ is uniformly continuous on I.

SOLUTION. Let $\varepsilon>0$. If $k=0$, any $\delta>0$ would work. So assume $k\neq 0$. There exists $\delta>0$ such that $|f(x)-f(y)|<\varepsilon/|k|$ whenever $|x-y|<\delta$. Then whenever $|x-y|<\delta$, we have

$$|k\cdot f(x)-k\cdot f(y)| = |k||f(x)-f(y)| < |k|\times\frac{\varepsilon}{|k|} = \varepsilon.$$

4.50: Prove that if f is uniformly continuous on a bounded interval I then f is bounded on I.

SOLUTION. Pick $\varepsilon = 1$. Then there exists $\delta > 0$ such that $|f(x) - f(y)| < 1$ whenever $|x - y| < \delta$. Now since I is bounded, we may pick points $y_1, \ldots, y_n \in I$ so that

$$I \subseteq \bigcup_{k=1}^{n} N_\delta(y_k).$$

Since $|f(x) - f(y_k)| < 1$ whenever $x \in N_\delta(y_k)$ for $k = 1, \ldots, n$, using triangle inequality, we get $|f(x)| < |f(y_k)| + 1$ whenever $x \in N_\delta(y_k)$ for $k = 1, \ldots, n$. Hence

$$|f(x)| < \max\{|f(y_k)| + 1 \mid k = 1, \ldots, n\}$$

for all $x \in I$. Therefore f is bounded on I.

4.54: Show by example that a continuous, bounded function on the bounded, open interval (a, b) need not be uniformly continuous on (a, b).
SOLUTION. Consider the function $f(x) = \sin(1/x)$ on the open interval $(0, 1)$. Note that $|f(x)| \leq 1$ for all $x \in (0, 1)$ and so is bounded. Also since $1/x$ is continuous on $(0, 1)$ and sin is continuous on $(1, \infty)$, f is continuous on $(0, 1)$ by Theorem **4.3**. However f is not uniformly continuous on $(0, 1)$ since

$$\left| f\left(\frac{1}{2n\pi}\right) - f\left(\frac{1}{2n\pi + \pi/2}\right) \right| = 1$$

for all $n \in \mathbb{N}$ (pick $\varepsilon < 1$ and note that $|1/2n\pi - 1/(2n\pi + \pi/2)| < \delta$ for any $\delta > 0$ when n is large enough).

6.5: Prove that if f is bounded on $[a, b]$ and has exactly one discontinuity in $[a, b]$ then f is Riemann-integrable on $[a, b]$.
SOLUTION. Suppose $c \in (a, b)$. Let $\varepsilon > 0$. Since f is bounded, there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a, b]$. Let $\delta = \varepsilon/12M$. Since f is continuous on $[a, c-\delta]$ and $[c+\delta, b]$, f is Riemann-integrable on $[a, c-\delta]$ and $[c+\delta, b]$ by Theorem **6.2**, so there exists partitions $P_1 = \{a = x_0 < \cdots < x_n = c - \delta\}$ of $[a, c - \delta]$ and $P_2 = \{c + \delta = y_0 < \cdots < y_m = c - \delta\}$ of $[c + \delta, b]$ such that

$$U(P_1, f) - L(P_1, f) < \frac{\varepsilon}{3} \quad \text{and} \quad U(P_2, f) - L(P_2, f) < \frac{\varepsilon}{3}.$$

Consider the partition of $[a, b]$ given by $P = P_1 \cup P_2$. Then

$$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i + 2\delta \sup_{x \in [c-\delta, c+\delta]} f(x) + \sum_{j=1}^{m} M_j \Delta y_j \leq U(P_1, f) + 2M\delta + U(P_2, f),$$

$$L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i + 2\delta \inf_{x \in [c-\delta, c+\delta]} f(x) + \sum_{j=1}^{m} m_j \Delta y_j \geq L(P_1, f) - 2M\delta + L(P_2, f),$$

and so

$$\begin{aligned} U(P, f) - L(P, f) &\leq [U(P_1, f) - L(P_1, f)] + 4M\delta + [U(P_2, f) - L(P_2, f)] \\ &< \frac{\varepsilon}{3} + 4M \times \frac{\varepsilon}{12M} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Hence f is Riemann integrable by Theorem **6.1**.

If $c = a$, a similar argument applied with a choice of $\delta = \varepsilon/4M$ and a partition P_2 of the interval $[a + \delta, b]$ shows that

$$U(P_2, f) - L(P_2, f) < \frac{\varepsilon}{2}$$

and that the partition of $[a,b]$ given by $P = \{a\} \cup P_2$ would have

$$U(P,f) - L(P,f) \le 2M\delta + [U(P_2,f) - L(P_2,f)]$$
$$< 2M \times \frac{\varepsilon}{4M} + \frac{\varepsilon}{2} = \varepsilon.$$

If $c = b$, a similar argument applied with a choice of $\delta = \varepsilon/4M$ and a partition P_1 of the interval $[a, b-\delta]$ shows that

$$U(P_1,f) - L(P_1,f) < \frac{\varepsilon}{2}$$

and that the partition of $[a,b]$ given by $P = P_1 \cup \{b\}$ would have

$$U(P,f) - L(P,f) \le [U(P_1,f) - L(P_1,f)] + 2M\delta$$
$$< 2M \times \frac{\varepsilon}{4M} + \frac{\varepsilon}{2} = \varepsilon.$$

6.6: Prove that if f is bounded on $[a,b]$ and f has only finitely many discontinuities in $[a,b]$ then f is Riemann-integrable on $[a,b]$.

Solution. Since we have not proved Theorem **6.8** in this section, we will prove it without assuming Theorem **6.8**. Let $c_1 < \cdots < c_k$ be all the discontinuities of f in $[a,b]$. We will assume that $a < c_1$ and $c_k < b$. The cases $a = c_1$ ro $c_k = b$ can be dealt with as in Problem **6.5**. The proof here is also similar to Problem **6.5**. Let $\varepsilon > 0$. Since f is bounded, there exists $M > 0$ such that $|f(x)| \le M$ for all $x \in [a,b]$. Let $\delta = \varepsilon/8Mk$. Since f is continuous on $[a, c_1-\delta], [c_1+\delta, c_2-\delta], \ldots, [c_k+\delta, b]$, f is Riemann-integrable on these intervals by Theorem **6.2**, so there exists partitions $P_0 = \{a = x_0 < \cdots < x_n = c_1 - \delta\}$ of $[a, c_1-\delta]$, $P_1 = \{c_1 + \delta = y_0 < \cdots < y_m = c_2 - \delta\}$ of $[c_1+\delta, c_2-\delta], \ldots, P_k = \{c_k + \delta = z_0 < \cdots < z_l = b\}$ of $[c_k+\delta, b]$ such that

$$U(P_i,f) - L(P_i,f) < \frac{\varepsilon}{2(k+1)} \quad \text{for } i = 0, \ldots, k.$$

Consider the partition of $[a,b]$ given by $P = P_0 \cup P_1 \cup \cdots \cup P_k$. Then

$$U(P,f) \le U(P_0,f) + 2M\delta + U(P_1,f) + 2M\delta + \cdots + 2M\delta + U(P_k,f) = 2Mk\delta + \sum_{i=0}^{k} U(P_i,f),$$

$$L(P,f) \ge L(P_0,f) - 2M\delta + L(P_1,f) - 2M\delta + \cdots - 2M\delta + L(P_k,f) = -2Mk\delta + \sum_{i=0}^{k} L(P_i,f),$$

and so

$$U(P,f) - L(P,f) \le 4Mk\delta + \sum_{i=0}^{k} [U(P_i,f) - L(P_i,f)$$
$$< 4Mk \times \frac{\varepsilon}{8Mk} + (k+1) \times \frac{\varepsilon}{2(k+1)} = \varepsilon.$$

Hence f is Riemann integrable by Theorem **6.1**.