5.2: Let \( f(x) = |x| \); show that

\[
f''(x) = \begin{cases} 
2 & \text{if } x > 0, \\
-2 & \text{if } x < 0, 
\end{cases}
\]

and that 0 is not in the domain of \( f''(x) \).

**Solution.** By Example 5.1, \( f'(x) = 2|x| \) for all \( x \in \mathbb{R} \). Let \( a > 0 \). Then \( f'(a) = 2a \) and \( f'(x) = 2x \) for all \( x \in \mathbb{N}^\delta(a) \) for some \( \delta > 0 \) (e.g. \( \delta = a/2 \)). So

\[
f''(a) = \lim_{x \to a} \frac{f'(x) - f'(a)}{x - a} = \lim_{x \to a} \frac{2x - 2a}{x - a} = \lim_{x \to a} 2 = 2.
\]

Let \( a < 0 \). Then \( f'(a) = -2a \) and \( f'(x) = -2x \) for all \( x \in \mathbb{N}^\delta(a) \) for some \( \delta > 0 \) (e.g. \( \delta = -a/2 \)). So

\[
f''(a) = \lim_{x \to a} \frac{f'(x) - f'(a)}{x - a} = \lim_{x \to a} \frac{-2x + 2a}{x - a} = \lim_{x \to a} -2 = -2.
\]

Since these are true for arbitrary \( a > 0 \) and \( a < 0 \), we have

\[
f''(x) = \begin{cases} 
2 & \text{if } x > 0, \\
-2 & \text{if } x < 0. 
\end{cases}
\]

Note that

\[
\lim_{x \to 0^+} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x \to 0^+} \frac{2x - 0}{x - 0} = 2,
\]

\[
\lim_{x \to 0^-} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x \to 0^-} \frac{-2x - 0}{x - 0} = -2,
\]

and so

\[
f''(0) = \lim_{x \to 0} \frac{f'(x) - f'(0)}{x - 0}
\]

does not exist.

5.5: Show that the following function is continuous at \( x = 0 \) but \( f'(0) \) does not exist. Find \( f'(x) \) for \( x \neq 0 \).

\[
f(x) = \begin{cases} 
x \sin \frac{1}{x} & \text{if } x \neq 0, \\
0 & \text{if } x = 0.
\end{cases}
\]

**Solution.** Recall that \( f \) is continuous by Example 4.6 everywhere on \( \mathbb{R} \). However

\[
f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x \sin(1/x) - 0}{x - 0} = \lim_{x \to 0} \frac{\sin(1/x)}{x}
\]

\[= \text{undefined.} \]
does not exist by Example 4.5. For any \( a \neq 0 \), \( f(x) = x \sin(1/x) \) for all \( x \in N_\delta^*(a) \) for some \( \delta > 0 \) (e.g. \( \delta = |a|/2 \)). So by product rule and chain rule,

\[
f'(a) = \frac{d}{dx} \left( x \sin \left( \frac{1}{x} \right) \right) \bigg|_{x=a} = \sin \frac{1}{x} - x \cos \frac{1}{x} \left( \frac{-1}{x^2} \right) \bigg|_{x=a} = \sin \frac{1}{a} - \frac{1}{a} \cos \frac{1}{a}.
\]

### 5.6: Show that the following function is differentiable for every \( x \in \mathbb{R} \) but \( f' \) is not continuous at \( x = 0 \).

\[
f(x) = \begin{cases} 
  x^2 \sin \frac{1}{x} & \text{if } x \neq 0, \\
  0 & \text{if } x = 0.
\end{cases}
\]

**Solution.** For any \( a \neq 0 \), the functions \( x^2 \), \( \sin x \), \( 1/x \) are all differentiable at \( x = a \) and so \( f \) is differentiable by product rule and chain rule. In fact,

\[
f'(a) = \frac{d}{dx} \left( x^2 \sin \frac{1}{x} \right) \bigg|_{x=a} = 2x \sin \frac{1}{x} - x^2 \cos \frac{1}{x} \left( \frac{-1}{x^2} \right) \bigg|_{x=a} = 2a \sin \frac{1}{a} - \cos \frac{1}{a}.
\]

For \( a = 0 \),

\[
f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin(1/x) - 0}{x - 0} = \lim_{x \to 0} x \sin \frac{1}{x} = 0.
\]

However

\[
\lim_{x \to 0} f'(x) = \lim_{x \to 0} \left[ 2x \sin \frac{1}{x} - \cos \frac{1}{x} \right]
\]

does not exist by Theorem 3.4 since \( \lim_{x \to 0} \cos(1/x) \) does not exist but \( \lim_{x \to 0} 2x \sin(1/x) \) does exist.

### 5.7: Show that the following function is differentiable for every \( x \in \mathbb{R} \) but \( f' \) is not unbounded at \( x = 0 \).

\[
f(x) = \begin{cases} 
  x^2 \sin \frac{1}{x^2} & \text{if } x \neq 0, \\
  0 & \text{if } x = 0.
\end{cases}
\]

**Solution.** For any \( a \neq 0 \), the functions \( x^2 \), \( \sin x \), \( 1/x^2 \) are all differentiable at \( x = a \) and so \( f \) is differentiable by product rule and chain rule. In fact,

\[
f'(a) = \frac{d}{dx} \left( x^2 \sin \frac{1}{x^2} \right) \bigg|_{x=a} = 2x \sin \frac{1}{x^2} - x^2 \cos \frac{1}{x^2} \left( \frac{-2}{x^3} \right) \bigg|_{x=a} = 2a \sin \frac{1}{a^2} - \frac{1}{a} \cos \frac{1}{a^2}.
\]

Let \( x_n = 1/\sqrt{2n\pi} \). Then \( \lim_{n \to \infty} x_n = 0 \) and

\[
\lim_{n \to \infty} f'(x_n) = \lim_{n \to \infty} \left[ \frac{2}{\sqrt{2n\pi}} \sin 2n\pi - \frac{1}{\sqrt{2n\pi}} \cos 2n\pi \right] = -\lim_{n \to \infty} \frac{1}{\sqrt{2n\pi}} = -\infty.
\]

So \( f' \) is unbounded at \( 0 \).

### 5.9: Let \( f(x) = e^{-|x|} \) for every \( x \in \mathbb{R} \). Is \( f \) continuous at \( x = 0? \) Differentiable at \( x = 0? \)

**Solution.** \( f \) is clearly continuous at \( x = 0 \) since \( \exp(-x) \) and \( |x| \) are both continuous at \( x = 0 \). However

\[
f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{e^{-|x|} - 0}{x - 0}
\]

does not exist since

\[
\lim_{x \to 0^+} \frac{e^{-|x|}}{x} = \lim_{x \to 0^+} \frac{e^{-x}}{x} = \infty.
\]
5.11: (a) Define

\[
f(x) = \begin{cases} 
\frac{1}{4^n} & \text{if } x = \frac{1}{2^n}, \ n = 1, 2, 3, \ldots, \\
0 & \text{otherwise.}
\end{cases}
\]

Is \( f \) differentiable at \( x = 0 \)? Verify.

**Solution.** We claim that \( f'(0) = 0 \). Let \((x_n)_{n\in\mathbb{N}}\) be a sequence such that \( \lim_{n\to\infty} x_n = 0 \) and \( x_n \neq 0 \) for all \( n \in \mathbb{N} \). Then

\[
\lim_{x\to0} \frac{f(x) - f(0)}{x - 0} = \begin{cases} 
\frac{1/4^n - 0}{1/2^n - 0} & \text{if } x_n = \frac{1}{2^n} \text{ for some } n \in \mathbb{N}, \\
\frac{0 - 0}{1/2^n - 0} & \text{otherwise,}
\end{cases}
\]

\[
= \begin{cases} 
\frac{1}{2^n} & \text{if } x_n = \frac{1}{2^n} \text{ for some } n \in \mathbb{N}, \\
0 & \text{otherwise,}
\end{cases}
\]

\[
\leq \frac{1}{2^n} \to 0
\]
as \( n \to \infty \). So by Exercise 3.26 and Theorem 3.6,

\[
\lim_{x\to0} \frac{f(x) - f(0)}{x - 0} = 0.
\]

(b) Define

\[
g(x) = \begin{cases} 
\frac{1}{2^n+1} & \text{if } x = \frac{1}{2^n}, \ n = 1, 2, 3, \ldots, \\
0 & \text{otherwise.}
\end{cases}
\]

Is \( g \) differentiable at \( x = 0 \)? Verify.

**Solution.** Note that

\[
\lim_{n\to\infty} \frac{g(1/2^n) - g(0)}{1/2^n - 0} = \lim_{n\to\infty} \frac{1/2^{n+1} - 0}{1/2^n - 0} = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2}
\]

but

\[
\lim_{n\to\infty} \frac{g(1/3^n) - g(0)}{1/3^n - 0} = \lim_{n\to\infty} \frac{0 - 0}{1/2^n - 0} = \lim_{n\to\infty} 0 = 0.
\]

By Theorem 3.6 and Theorem 3.6, the limit

\[
\lim_{x\to0} \frac{g(x) - g(0)}{x - 0}
\]

does not exist. So \( g \) is not differentiable at \( x = 0 \).

5.12: Prove that if \( x = g(t) \), \( y = f(t) \) are differentiable in some neighborhood of \( t_0 \), \( g'(t_0) \neq 0 \) and \( x_0 = g(t_0) \), \( y_0 = f(t_0) \) and if \( y = H(x) \) in some neighborhood of \( x_0 \) then

\[
\frac{dy}{dx}_{x_0} = \frac{f'(t_0)}{g'(t_0)} = \left. \frac{dy}{dt} \right|_{t_0} / \left. \frac{dx}{dt} \right|_{t_0}.
\]

**Solution.** Note that \( y = f(t) = H(g(t)) \) in some neighborhood of \( x_0 \). So by chain rule,

\[
f'(t_0) = H'(g(t_0))g'(t_0) = H'(x_0)g'(t_0).
\]

Since \( g'(t_0) \neq 0 \), we may write

\[
H'(x_0) = \frac{f'(t_0)}{g'(t_0)}.
\]
In other words,

\[
\frac{dy}{dx}\bigg|_{x_0} = \frac{dy/dt|_{t_0}}{dx/dt|_{t_0}}.
\]

5.14: Prove that if \( f \) is differentiable on \((a,b)\) and \( f'(x) \leq 0 \) for every \( x \in (a,b) \) then \( f \) is monotone decreasing on \((a,b)\).

**SOLUTION.** Apply Theorem 5.7 to \(-f\).

5.16: Suppose \( f'(x) = g'(x) \) for all \( x \) in some interval \( I \). Prove that there exists some constant \( k \) such that \( f(x) = g(x) + k \) for all \( x \in I \).

**SOLUTION.** Apply Exercise 5.15 to \( f - g \).

5.19: Prove that for any real number \( b \) the polynomial \( f(x) = x^3 + x + b \) has exactly one real root; that is, there exists a unique \( x_0 \in \mathbb{R} \) such that \( f(x_0) = 0 \).

**SOLUTION.** If \( b = 0 \), then clearly \( x_0 = 0 \) is a real root. If \( b \neq 0 \),

\[
f(b)f(-b) = -b^6 - 2b^4 < 0,
\]

and so \( f(b) \) and \( f(-b) \) are of opposite signs. Since \( f \) is continuous, the intermediate value theorem says that there exists at least one \( x_0 \) between \( b \) and \(-b \) such that \( f(x_0) = 0 \). So \( f \) has at least one real root for any \( b \in \mathbb{R} \). If \( x_1 \neq x_0 \) is another root of \( f \). Then \( f(x_0) = 0 = f(x_1) \) and since \( f \) is differentiable, the mean value theorem says that there exists a \( y \) such that \( f'(y) = 0 \). This contradicts

\[
f'(x) = 3x^2 + 1 > 0 \quad \text{for all } x \in \mathbb{R}.
\]