MATH 104: INTRODUCTORY ANALYSIS
SPRING 2009/10
PROBLEM SET 6 SOLUTIONS

5.2: Let $f(x) = x|x|$; show that

$$f''(x) = \begin{cases} 2 & \text{if } x > 0, \\ -2 & \text{if } x < 0, \end{cases}$$

and that 0 is not in the domain of $f''(x)$.

Solution. By Example **5.1**, $f'(x) = 2|x|$ for all $x \in \mathbb{R}$. Let $a > 0$. Then $f'(a) = 2a$ and $f'(x) = 2x$ for all $x \in N_\delta^*(a)$ for some $\delta > 0$ (e.g. $\delta = a/2$). So

$$f''(a) = \lim_{x\to a} \frac{f'(x) - f'(a)}{x - a} = \lim_{x\to a} \frac{2x - 2a}{x - a} = \lim_{x\to a} 2 = 2.$$

Let $a < 0$. Then $f'(a) = -2a$ and $f'(x) = -2x$ for all $x \in N_\delta^*(a)$ for some $\delta > 0$ (e.g. $\delta = -a/2$). So

$$f''(a) = \lim_{x\to a} \frac{f'(x) - f'(a)}{x - a} = \lim_{x\to a} \frac{-2x + 2a}{x - a} = \lim_{x\to a} -2 = -2.$$

Since these are true for arbitrary $a > 0$ and $a < 0$, we have

$$f''(x) = \begin{cases} 2 & \text{if } x > 0, \\ -2 & \text{if } x < 0. \end{cases}$$

Note that

$$\lim_{x\to 0^+} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x\to 0^+} \frac{2x - 0}{x - 0} = 2,$$
$$\lim_{x\to 0^-} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x\to 0^-} \frac{-2x - 0}{x - 0} = -2,$$

and so

$$f''(0) = \lim_{x\to 0} \frac{f'(x) - f'(0)}{x - 0}$$

does not exist.

5.5: Show that the following function is continuous at $x = 0$ but $f'(0)$ does not exist. Find $f'(x)$ for $x \neq 0$.

$$f(x) = \begin{cases} x \sin \dfrac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Solution. Recall that f is continuous by Example **4.6** everywhere on $\mathbb{R}$. However

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x\sin(1/x) - 0}{x - 0} = \lim_{x\to 0} \sin\frac{1}{x}$$

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does not exist by Example **4.5**. For any $a \neq 0$, $f(x) = x\sin(1/x)$ for all $x \in N^*_\delta(a)$ for some $\delta > 0$ (e.g. $\delta = |a|/2$). So by product rule and chain rule,

$$f'(a) = \frac{d}{dx}\left(x\sin\frac{1}{x}\right)\bigg|_{x=a} = \sin\frac{1}{x} - x\cos\frac{1}{x} \times \frac{1}{x^2}\bigg|_{x=a} = \sin\frac{1}{a} - \frac{1}{a}\cos\frac{1}{a}.$$

5.6: Show that the following function is differentiable for every $x \in \mathbb{R}$ but f' is not continuous at $x = 0$.

$$f(x) = \begin{cases} x^2 \sin\dfrac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Solution. For any $a \neq 0$, the functions x^2, $\sin x$, $1/x$ are all differentiable at $x = a$ and so f is differentiable by product rule and chain rule. In fact,

$$f'(a) = \frac{d}{dx}\left(x^2\sin\frac{1}{x}\right)\bigg|_{x=a} = 2x\sin\frac{1}{x} - x^2\cos\frac{1}{x} \times \frac{1}{x^2}\bigg|_{x=a} = 2a\sin\frac{1}{a} - \cos\frac{1}{a}.$$

For $a = 0$,

$$f'(0) = \lim_{x\to 0}\frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0}\frac{x^2\sin(1/x) - 0}{x - 0} = \lim_{x\to 0} x\sin\frac{1}{x} = 0.$$

However

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0}\left[2x\sin\frac{1}{x} - \cos\frac{1}{x}\right]$$

does not exist by Theorem **3.4** since $\lim_{x\to 0}\cos(1/x)$ does not exist but $\lim_{x\to 0} 2x\sin(1/x)$ does exist.

5.7: Show that the following function is differentiable for every $x \in \mathbb{R}$ but f' is not unbounded at $x = 0$.

$$f(x) = \begin{cases} x^2 \sin\dfrac{1}{x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Solution. For any $a \neq 0$, the functions x^2, $\sin x$, $1/x^2$ are all differentiable at $x = a$ and so f is differentiable by product rule and chain rule. In fact,

$$f'(a) = \frac{d}{dx}\left(x^2\sin\frac{1}{x^2}\right)\bigg|_{x=a} = 2x\sin\frac{1}{x^2} - 2x^2\cos\frac{1}{x^2} \times \frac{1}{x^3}\bigg|_{x=a} = 2a\sin\frac{1}{a^2} - \frac{1}{a}\cos\frac{1}{a^2}.$$

Let $x_n = 1/\sqrt{2n\pi}$. Then $\lim_{n\to\infty} x_n = 0$ and

$$\lim_{n\to\infty} f'(x_n) = \lim_{n\to\infty}\left[\frac{2}{\sqrt{2n\pi}}\sin 2n\pi - \frac{1}{\sqrt{2n\pi}}\cos 2n\pi\right] = -\lim_{n\to\infty}\frac{1}{\sqrt{2n\pi}} = -\infty.$$

So f is unbounded at 0.

5.9: Let $f(x) = e^{-|x|}$ for every $x \in \mathbb{R}$. Is f continuous at $x = 0$? Differentiable at $x = 0$?
Solution. f is clearly continuous at $x = 0$ since $\exp(-x)$ and $|x|$ are both continuous at $x = 0$. However

$$f'(0) = \lim_{x\to 0}\frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0}\frac{e^{-|x|} - 0}{x - 0}$$

does not exist since

$$\lim_{x\to 0^+}\frac{e^{-|x|}}{x} = \lim_{x\to 0^+}\frac{e^{-x}}{x} = \infty.$$

5.11: (a) Define

$$f(x) = \begin{cases} \dfrac{1}{4^n} & \text{if } x = \dfrac{1}{2^n}, \quad n = 1, 2, 3, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Is f differentiable at $x = 0$? Verify.

Solution. We claim that $f'(0) = 0$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence such that $\lim_{n\to\infty} x_n = 0$ and $x_n \neq 0$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned} \left| \frac{f(x_n) - f(0)}{x_n - 0} \right| &= \begin{cases} \dfrac{1/4^n - 0}{1/2^n - 0} & \text{if } x_n = \dfrac{1}{2^n} \text{ for some } n \in \mathbb{N}, \\ \dfrac{0 - 0}{1/2^n - 0} & \text{otherwise,} \end{cases} \\ &= \begin{cases} \dfrac{1}{2^n} & \text{if } x_n = \dfrac{1}{2^n} \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise,} \end{cases} \\ &\leq \frac{1}{2^n} \to 0 \end{aligned}$$

as $n \to \infty$. So by Exercise **3.26** and Theorem **3.6**,

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = 0.$$

(b) Define

$$g(x) = \begin{cases} \dfrac{1}{2^{n+1}} & \text{if } x = \dfrac{1}{2^n}, \quad n = 1, 2, 3, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Is g differentiable at $x = 0$? Verify.

Solution. Note that

$$\lim_{n\to\infty} \frac{g(1/2^n) - g(0)}{1/2^n - 0} = \lim_{n\to\infty} \frac{1/2^{n+1} - 0}{1/2^n - 0} = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2}$$

but

$$\lim_{n\to\infty} \frac{g(1/3^n) - g(0)}{1/3^n - 0} = \lim_{n\to\infty} \frac{0 - 0}{1/2^n - 0} = \lim_{n\to\infty} 0 = 0.$$

By Theorem **3.6** and Theorem **3.6**, the limit

$$\lim_{x\to 0} \frac{g(x) - g(0)}{x - 0}$$

does not exist. So g is not differentiable at $x = 0$.

5.12: Prove that if $x = g(t)$, $y = f(t)$ are differentiable in some neighborhood of t_0, $g'(t_0) \neq 0$ and $x_0 = g(t_0)$, $y_0 = f(t_0)$ and if $y = H(x)$ in some neighborhood of x_0 then

$$\left.\frac{dy}{dx}\right|_{x_0} = \frac{f'(t_0)}{g'(t_0)} = \frac{dy/dt|_{t_0}}{dx/dt|_{t_0}}.$$

Solution. Note that $y = f(t) = H(g(t))$ in some neighborhood of x_0. So by chain rule,

$$f'(t_0) = H'(g(t_0))g'(t_0) = H'(x_0)g'(t_0).$$

Since $g'(t_0) \neq 0$, we may write

$$H'(x_0) = \frac{f'(t_0)}{g'(t_0)}.$$

In other words,

$$\left.\frac{dy}{dx}\right|_{x_0} = \frac{dy/dt|_{t_0}}{dx/dt|_{t_0}}.$$

5.14: Prove that if f is differentiable on (a, b) and $f'(x) \leq 0$ for every $x \in (a, b)$ then f is monotone decreasing on (a, b).
SOLUTION. Apply Theorem **5.7** to $-f$.

5.16: Suppose $f'(x) = g'(x)$ for all x in some interval I. Prove that there exists some constant k such that $f(x) = g(x) + k$ for all $x \in I$.
SOLUTION. Apply Exercise **5.15** to $f - g$.

5.19: Prove that for any real number b the polynomial $f(x) = x^3 + x + b$ has exactly one real root; that is, there exists a unique $x_0 \in \mathbb{R}$ such that $f(x_0) = 0$.
SOLUTION. If $b = 0$, then clearly $x_0 = 0$ is a real root. If $b \neq 0$,

$$f(b)f(-b) = -b^6 - 2b^4 < 0,$$

and so $f(b)$ and $f(-b)$ are of opposite signs. Since f is continuous, the intermediate value theorem says that there exists at least one x_0 between b and $-b$ such that $f(x_0) = 0$. So f has at least one real root for any $b \in \mathbb{R}$. If $x_1 \neq x_0$ is another root of f. Then $f(x_0) = 0 = f(x_1)$ and since f is differentiable, the mean value theorem says that there exists a y such that $f'(y) = 0$. This contradicts

$$f'(x) = 3x^2 + 1 > 0 \quad \text{for all } x \in \mathbb{R}.$$