# MATH 104: INTRODUCTORY ANALYSIS
# SPRING 2009/10
# PROBLEM SET 5 SOLUTIONS

**4.4:** Prove that if $f$ is continuous at $x_0$ and $g$ is discontinuous at $x_0$ then $f+g$ must have a discontinuity at $x_0$.
SOLUTION. Suppose not and $f+g$ is continuous at $x_0$. Then $g=(f+g)-f$ must be continuous at $x_0$ by Theorem **4.2**, a contradiction.

**4.6:** Show that $f\cdot g$ can be continuous at $x_0$ even though both $f$ and $g$ have discontinuities at $x_0$
SOLUTION. Let

$$f(x)=\begin{cases}1 & \text{if } x\geq 0,\\ 0 & \text{if } x<0,\end{cases}\qquad g(x)=\begin{cases}0 & \text{if } x\geq 0,\\ 1 & \text{if } x<0.\end{cases}$$

Then $f$ and $g$ have a jump discontinuity at 0. $f\cdot g(x)=f(x)g(x)=0$ for all $x\in\mathbb{R}$. So $f\cdot g$ is a constant function and is continuous on $\mathbb{R}$ by Exercise **3.17**.

**4.8:** Show that the composition function $g\circ f$ can be continuous at $x_0$ even though $f$ or $g$ or both $f$ and $g$ are discontinuous at $x_0$.
SOLUTION. Consider the same $f$ and $g$ in Problem **4.6**. Then $g\circ f(x)=g(f(x))=0$ for all $x\in\mathbb{R}$ since $g(1)=g(0)=0$. So $g\circ f$ is a constant function and is continuous on $\mathbb{R}$ by Exercise **3.17**.

**4.9:** Prove that the function

$$f(x)=\begin{cases}x & \text{if } x\in\mathbb{Q},\\ 0 & \text{if } x\notin\mathbb{Q},\end{cases}$$

has a discontinuity of the second kind at each nonzero real number.
SOLUTION. Let $a\in\mathbb{R}$ and $a\neq 0$. Note that for any $n\in\mathbb{N}$, the interval $(a,a+1/n)$ must contain a rational number $x_n$ and an irrational number $y_n$ by Theorems **1.9** and **1.10**. Note that $a<x_n<a+1/n$ and $a<y_n<a+1/n$, so by Squeezing Lemma,

$$\lim_{n\to\infty}x_n=a=\lim_{n\to\infty}y_n.$$

If $\lim_{x\to a^+}f(x)$ exists, then by Exercise **3.36**,

$$\lim_{x\to a^+}f(x)=\lim_{n\to\infty}f(x_n)=\lim_{n\to\infty}x_n=a,$$

and also

$$\lim_{x\to a^+}f(x)=\lim_{n\to\infty}f(y_n)=\lim_{n\to\infty}0=0.$$

Since $a\neq 0$, $\lim_{x\to a^+}f(x)$ does not exist. Hence $f$ has a Type II discontinuity at $a$.

**4.11:** Find a function $f$ which has a discontinuity of the second kind at every real number although $f\circ f$ is continuous on $\mathbb{R}$.

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SOLUTION. By Example **4.4**, the function

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ \\ 0 & \text{if } x \notin \mathbb{Q}, \end{cases}$$

has a Type II discontinuity at every $x \in \mathbb{R}$. However $f \circ f(x) = f(f(x)) = 1$ for every $x \in \mathbb{R}$ since $f(x) \in \mathbb{Q}$ for every $x \in \mathbb{R}$. So $f \circ f$ is a constant function and is continuous on $\mathbb{R}$ by Exercise **3.17**.

**4.13:** Prove that if $f$ and $g$ are each continuous on $(a, b)$ and $f(x) = g(x)$ for every rational $x \in (a, b)$ then $f(x) = g(x)$ for every $x \in (a, b)$.
SOLUTION. Let $x \in (a, b)$. If $x \in \mathbb{Q}$, then $f(x) = g(x)$ and we are done. If $x \notin \mathbb{Q}$, then choose a sequence $x_n \in \mathbb{Q}$ converging to $x$ like in Problem **4.9**. Since $f(x_n) = g(x_n)$ for all $n \in \mathbb{N}$, by the continuity of $f$ and $g$ and Theorem **4.1**,

$$f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = g(x)$$

and we also done.

**4.14:** Prove: $f$ is right-continuous at $x_0$ if and only if $f(x_n) \to f(x_0)$ for every sequence $(x_n)_{n\in\mathbb{N}}$ in the domain of $f$ with $x_n \to x_0$ and $x_n \geq x_0$ for $n = 1, 2, 3, \ldots$.
SOLUTION. This follows from Exercise **3.36** with $L = f(x_0)$.

**4.17:** Find all functions $f$ which are continuous on $\mathbb{R}$ and which satisfy the equation $f(x)^2 = x^2$ for each $x \in \mathbb{R}$. *Hint*: There are four possible solutions.
SOLUTION. The four functions are given by $f_1(x) = x$, $f_2(x) = -x$, $f_3(x) = |x|$, $f_4(x) = -|x|$. Note that $f(x)^2 - x^2 = 0$ iff $(f(x) - x)(f(x) + x) = 0$ iff $f(x) = \pm x$ for all $x \in \mathbb{R}$. The only way $f$ could be continuous on $\mathbb{R}$ is if it is one of these four functions.

**4.18:** Prove that if $g$ is continuous at $x_0 = 0$, $g(0) = 0$ and for some $\delta > 0$, $|f(x)| \leq |g(x)|$ for each $x \in N_\delta(0)$ then $f$ is continuous at $x_0 = 0$.
SOLUTION. By Exercise **3.24** and the continuity of $g$ at 0, $\lim_{x\to 0}|g(x)| = 0$. Since $0 \leq |f(x)| \leq |g(x)|$ for each $x \in N_\delta(0)$, by Exercise **3.27**, we have $\lim_{x\to 0}|f(x)| = 0$. So by Exercise **3.26**, $\lim_{x\to 0} f(x) = 0$. But $|f(0)| \leq |g(0)| = 0$ implies that $f(0) = 0$. So $\lim_{x\to 0} f(x) = f(0)$.

**4.29:** Suppose $f$ is continuous on $\mathbb{R}$. Prove that if $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ both exist then $f$ is bounded on $\mathbb{R}$ but the converse does not hold.
SOLUTION. Let $\lim_{x\to\infty} f(x) = L_1$ and $\lim_{x\to-\infty} f(x) = L_2$. Pick $\varepsilon = 1$. Then there exists $K_1 > 0$ and $K_2 > 0$ such that $|f(x)-L_1| < 1$ whenever $x > K_1$ and $|f(x)-L_2| < 1$ whenever $x < -K_2$. Hence $|f(x)| < |L_1| + 1$ for all $x \in (K_1, \infty)$ and $|f(x)| < |L_2| + 1$ for all $x \in (-\infty, K_2)$. Since $f$ is continuous, it is bounded on $[-K_2, K_1]$ by Theorem **4.4**, so there exists $L_3 > 0$ such that $|f(x)| < L_3$ for all $x \in [-K_2, K_1]$. Let $M = \max(|L_1| + 1, |L_2| + 1, L_3)$. Then $|f(x)| < M$ for all $x \in \mathbb{R}$.

The converse is not true: Consider $f(x) = \sin x$. Then $f$ is continous and bounded on $\mathbb{R}$ (since $|\sin x| \leq 1$ for all $x \in \mathbb{R}$) but neither $\lim_{x\to\infty} f(x)$ nor $\lim_{x\to-\infty} f(x)$ exists by Exercise **3.52** since

$$\lim_{n\to\infty} f(2n\pi) = 0 \neq 1 = \lim_{n\to\infty} f\left((2n + \tfrac{1}{2})\pi\right),$$
$$\lim_{n\to\infty} f(-2n\pi) = 0 \neq 1 = \lim_{n\to\infty} f\left(-(2n + \tfrac{1}{2})\pi\right).$$

**4.30:** Verify that the function in Example **4.5** satisfies the intermediate-value property on $[-1, 1]$.

Solution. Let $x_1, x_2 \in [-1, 1]$ and $x_1 < x_2$. If $0 < x_1 < x_2$ or $x_1 < x_2 < 0$, then $\sin(1/x)$ is continuous on $[x_1, x_2]$ and so the intermediate-value property is satisfied by Theorem **4.6**. If $x_1 \leq 0 < x_2$ or $x_1 < 0 \leq x_2$, then $[x_1, x_2]$ contains either the interval $(0, x_2)$ or $(x_1, 0)$ (or both). The image of $(0, x_2)$ under $1/x$ is $(1/x_2, \infty)$ and the image of $(1/x_2, \infty)$ under $\sin(x)$ is $[-1, 1]$. Likewise, the image of $(x_1, 0)$ under $1/x$ is $(-\infty, 1/x_1)$ and the image of $(-\infty, 1/x_1)$ under $\sin(x)$ is $[-1, 1]$. Hence in either case the intermediate-value property must be satisfied.

**4.31:** Prove that if $f$ is continuous on any interval then $f$ satisfies the intermediate-value property on that interval.
Solution. Note that the interval $I$ could be of the form $(a, b)$, $(a, b]$, $[a, b)$, or $[a, b]$. So Theorem **4.6** does not necessarily apply to $I$ (it only applies in the last case). However, for any $x_1, x_2 \in I$ with $x_1 < x_2$, we may apply Theorem **4.6** to the closed interval $[x_1, x_2]$ and hence $f$ satisfies the intermediate-value property on $I$.

**4.33:** Prove that the polynomial $p(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$, where $a_0 \neq 0$ and $n$ is an odd natural number, has at least one real root; that is, there exists $x_0 \in \mathbb{R}$ such that $p(x_0) = 0$.
Solution. Note that if $a_0 > 0$ then $\lim_{x\to\infty} p(x) = \infty$ and $\lim_{x\to-\infty} p(x) = -\infty$ whereas if $a_0 < 0$ then $\lim_{x\to\infty} p(x) = -\infty$ and $\lim_{x\to-\infty} p(x) = \infty$. You can show these rigorously by emulating the proof in Exercise **3.8** in Homework **3**. WLOG, suppose $a_0 < 0$, then since $\lim_{x\to\infty} p(x) = -\infty$ and $\lim_{x\to-\infty} p(x) = \infty$, pick $M = 1$ and we know that there exists $K_1 > 0$ such that $p(x) < -1$ whenever $x > K_1$ and $K_2 > 0$ such that $p(x) > 1$ whenever $x < -K_2$. Hence in particular $p(K_1 + 1) < 0$ and $p(-K_2 - 1) > 0$. Since $p$ is continuous, by Baby IVT, it has a root $x_0$. Likewise for the case $a_0 > 0$.

**4.37:** Prove that the function $f(x) = x^3 + x^2 - 3x - 3$ has a root between 1 and 2, between 1.5 and 2, between 1.5 and 1.75, between 1.625 and 1.75, etc. Note that if we continue this procedure we shall be able to approximate the root as closely as we wish.
Solution. $f(1) = -4$, $f(1.5) = -1.875$, $f(1.625) = -0.94336$, $f(1.75) = 0.17188$, $f(2) = 3$. Since $f(1)f(2) < 0$, $f(1.5)f(2) < 0$, $f(1.5)f(1.75) < 0$, $f(1.625)f(1.75) < 0$, and $f$ is continuous, by Baby IVT $f$ has a root between 1 and 2, between 1.5 and 2, between 1.5 and 1.75, between 1.625 and 1.75.

**4.40:** Prove that if $f$ is monotone on $[a, b]$ and $f$ satisfies the intermediate-value property on $[a, b]$ then $f$ is continuous on $[a, b]$.
Solution. We may assume WLOG that $f$ is monotone increasing. Let $x_0 \in (a, b)$. Recall the following shorthands that we introduced during the lectures

$$f(x_0^-) := \lim_{x\to x_0^-} f(x) \qquad \text{and} \qquad f(x_0^+) = \lim_{x\to x_0^+} f(x).$$

Since $f$ is monotone increasing,

$$\sup_{x\in[a,x_0)} f(x) = f(x_0^-) \leq f(x_0) \leq f(x_0^+) = \inf_{x\in(x_0,b]} f(x).$$

We claim that $f(x_0^-) = f(x_0) = f(x_0^+)$. Suppose not, say

$$f(x_0) < f(x_0^+).$$

Take $k$ such that

$$f(x_0) < k < f(x_0^+). \tag{1}$$

Note that $f(x_0^+) \leq f(b)$ by monotonicity. So

$$f(x_0) < k < f(b).$$

Now given any $x \in (x_0, b)$, we have $f(x) \geq f(x_0^+)$ by monotonicity and so $f(x) > k$ by (1). In other words, there is no $x \in (x_0, b)$ such that $f(x) = k$, a contradiction to $f$ satisfying the intermediate-value property. Likewise we may show that $f(x_0) = f(x_0^-)$ and so $f$ is continuous at $x_0$.

**4.42:** Find a continuous function $f$ on $\mathbb{R}$ such that for every real number $c$, $f(x) = c$ has exactly three solutions.

Solution. Define the function $g : [-3, 3] \to \mathbb{R}$ by

$$g(x) = \begin{cases} x+2 & \text{if } -3 \leq x \leq -1, \\ -x & \text{if } -1 \leq x \leq 1, \\ x-2 & \text{if } 1 \leq x \leq 3, \end{cases}$$

and define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = g(x - 6n) + 2n \quad \text{for } 6n - 3 \leq x \leq 6n + 3,\ n \in \mathbb{Z}.$$

The function $f$ has the desired property.